OPTIMIZATION OF MULTIFACILITY PRODUCTION SCHEDULING

763

by

JAYAWANTH DATTATREYARAO BANTWAL

B.Sc Engineering Kerala University
Kerala, India, 1964

A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Industrial Engineering

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1967

Approved by

Ching-Lai Hwang
Major Professor

LD
2668
T4
1967
B219
c.2

TABLE OF CONTENTS

1. GENERAL INTRODUCTION

The objective of this report is to present the study made by Zangwill (8, 9) dealing with a deterministic multiproduct multifacility production planning and inventory model and that by Nelson (7) concerned with labour assignment as a dynamic control problem. The optimization technique employed by Zangwill in arriving at an optimal production schedule is the well known dynamic programming. In this report, the same solution has been obtained more simply by a discrete version of the maximum principle. Nelson has optimized the labour assignment in a labor and machine limited production system by the continuous maximum principle. In this report the same problem for discrete time intervals is studied by the discrete maximum principle.

The deterministic multiproduct, multifacility, multiperiod production planning model developed by Zangwill is essentially a linking together of several single facility models; the linking is arranged to form a multifacility acyclic network. This type of linking preserves many of the interdependences in both production and cost that are inherent in multiproduct, multifacility production systems. An example of such interdependence is a situation in which a facility cannot produce before it receives inputs from another facility. In addition, the acyclic network also implies many multiproduct and subassembly situations.

The model considers concave production cost functions which may depend on production in several different facilities and piecewise concave inventory cost functions. The optimization problem consists in determining an optimal production schedule that specifies how much each facility in the network should produce in each period for the next n periods so that the total production and inventory cost is minimized.

First some basic definitions are given and the problem is stated clearly with respect to the cost structure of the acyclic network. Next the parallel facility case is discussed in detail. An algorithm based on the dynamic programming employed by Zangwill is presented and applied in the solution of Example 1. The same example is then solved by the discrete maximum principle. This is followed by Example 2 in which the parallel facility case with a nonlinear cost function is discussed. This is treated as a multi-dimensional process and solved by the discrete maximum principle. The series facility case and the dynamic programming algorithm for obtaining the optimal solution are then presented. Example 3 which deals with 3 facilities in series is then solved by both dynamic programming and the discrete maximum principle. In Example 4, four facilities in series are considered. Example 5 is a mere extension of Example 4 with one additional facility which yields entirely different results from those of Example 4. Both Example 4 and Example 5 are solved by dynamic programming.

Next a multiperiod production planning model with a concave cost function and a backlog of demand is presented. Here again the technique of dynamic programming employed by Zangwill is presented first. Example 6, which illustrates the usefulness of this type of problem is solved by both techniques, namely, dynamic programming and the discrete maximum principle. In each of the above examples an attempt has been made to compare the efficiencies of the two techniques.

The last section of the report is devoted to the discussion of labour assignment as a dynamic control problem in a multifacility network. The system considered has L labourers and m machine centres. There are f_i identical machines in machine centre i, $i = 1, 2, \ldots, m$. The total number of labourers available is less than total number of machines. Thus labour

is a limited resource. The work pieces which arrive at the machine centre are processed on different machines in a definite order. The work piece which arrives at a particular machine centre in which all the machines or all those machines which have laboures assigned to them are already engaged in processing the work pieces which arrived before is forced to wait in a queue. This incurs a cost that is known as the in-process inventory cost. The problem at hand consists in seeking an optimal way in which labour is assigned to different machine centres so as to minimize the total in-process inventory cost during a given time period.

In the original model analyzed by Nelson (7), the work pieces are assumed to arrive at the machine centres at a continuous rate and hence the continuous maximum principle is employed in optimizing labour assignment. The model considered in this report assumes that the work pieces arrive at discrete time intervals and is solved by the discrete maximum principle. Example 7 is a simple numerical example which illustrates the applicability of the algorithm.

2. DEFINITIONS AND STATEMENT OF THE PROBLEM (8, 9)

2.1 BASIC FACILITY

The rudimentary building block in the acyclic network is the basic facility, or simply, facility. Each facility is as shown in Fig. 1 and consists of a production line and an inventory for the product.

The facility receives inputs from raw materials and one or more facilities and then in each period manufactures a specific product on its own production line. This product is then stored in inventory until needed either to satisfy market requirements for the product or to supply input to other facilities.

Let r_i^j, $r_i^j \geq 0$ be the market requirements for facility j's ($j = 1, 2, \ldots, N$) product in period i ($i = 1, 2, \ldots, n$), where n is the number of periods under consideration and there are N facilities. It is assumed that all requirements r_i^j are fixed and known in advance. Let x_i^j, $x_i^j \geq 0$ be the production completed in period i by facility j and I_i^j be the inventory at the end of period i, in facility j.

2.2 ACYCLIC NETWORK

The individual facilities are linked together to form an acyclic network as shown in Fig. 2. Each facility can receive inputs from either raw materials or from lower numbered facilities. Similarly each facility can supply only higher numbered facilities or market requirements for its own pr[illegible]t.

The inventory equations for the network express the condition that the inventory level in period i of facility j is the total amount of production completed in the facility through period i less the amount desired to satisfy market requirements and inputs to other facilities through

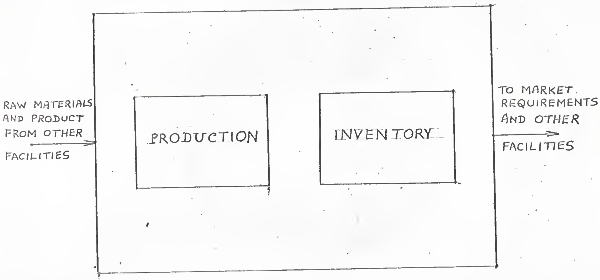


Fig. 1 THE BASIC FACILITY

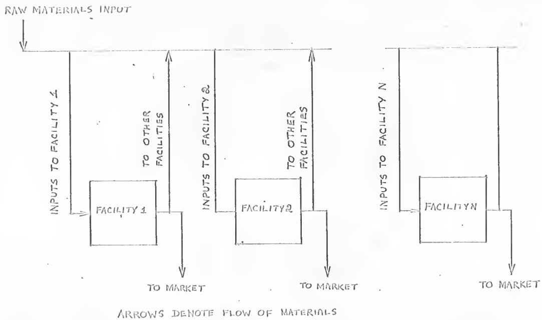


Fig 2. THE ACYCLIC NETWORK

period i. Let a^{jh} ($a^{jh} \geq 0$) be the number of units of facility j's product required to produce one unit of facility h's product. $a^{jh} = 0$ for $h \leq j$, since facility j supplies only higher numbered facilities. However, for any $j < h$, a^{jh} could be zero, if facility j does not supply inputs to facility h. It is assumed that there is no time loss in transmission of goods from one facility to another. However, facility j can have time lag in production. Let L_j be a non negative integer that represents the number of periods lag from the start of production in facility j until the completion of production. Production started in period i at facility j is thereby completed in period $i + L_j$. The amount desired out of facility j in period i as inputs to other facilities is therefore

$$\sum_{h=j+1}^{h=N} a^{jh} x_{i+L_h}^{h} .$$

The total demand on facility j in period i denoted by y_i^j is

$$y_i^j = r_i^j + \sum_{h=j+1}^{h=N} a^{jh} x_{i+L_h}^{h} . \tag{1}$$

Note that production can be started in each period so that at any instant of time there may be several batches of production started in a particular facility but not yet completed.

Production lags introduce a difficulty in that it might be impossible for certain facilities to complete production in time to satisfy some initial market requirements. This is an artificial difficulty imposed by considering a time horizon of n periods. There is no loss of generality in assuming that market requirements are zero in any period that cannot be supplied because of the production lags.

From the above discussion the inventory level equation becomes:

$$I_i^j = \sum_{h=1}^{h=i} (x_h^j - y_h^j) \tag{2}$$

for all i and j.

2.3 BACKLOGGING

It is assumed that each facility can backlog total demand for its product for a certain integral number of periods. Let α_j be a non negative integer denoting the number of periods of backlog permitted for facility j. Each facility can have a different α_j but for a particular facility α_j is fixed. For any given facility a backlog of up to α_j periods of total demand is permitted; but no more than α_j periods. α_j is called the backlog limit for facility j. If $\alpha_j = 0$ no backlogging is permitted in facility j. The inventory with backlogging is

$$I_i^j \geq - \sum_{h=i-\alpha_j+1}^{h=i} y_h^j . \tag{3}$$

Consider a situation in which facility j-1 supplies facility j so that $a^{j-1,j} > 0$. If $\alpha_{j-1} \geq 1$ equation (3) would permit production in facility j without receiving goods from facility j-1. The backlogging allows this to occur. In many production problems the above situation is meaningless because facility j could not produce without the inputs from the facility j-1. In such cases α_{j-1} must be zero. However, if there is a large buffer stock of facility j-1's product at facility j, then $\alpha_{j-1} \geq 1$ might be permissible.

The market requirements are known in advance.

2.4 THE MULTIPRODUCT MULTIFACILITY SYSTEM

The acyclic network can be used to model a wide variety of production and inventory systems as will be illustrated.

Fig. 3a depicts a multi-product system in which facility 1 produces a subassembly that is used to make one final product in facility 2 and a different final product in facility 3.

This system is modelled by using an acyclic network with 3 facilities, letting $a^{12} > 0$, $a^{13} > 0$ and setting $a^{23} = 0$.

Figure 3b exhibits another structure. Two different sub-assemblies; one produced in facility 1 and the other in facility 2 are required to make the final product in facility 3. This system can be modelled by using a 3 facility acyclic network with $a^{12} = 0$, $a^{13} > 0$ and $a^{23} > 0$.

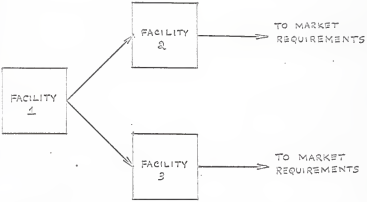
The acyclic network can clearly be applied to many other complex multi-product multi-facility systems, an example of which is given in Fig. 4.

NOTATION: Now let us define certain vectors. Let $x^j = (x_1^j, x_2^j, \ldots, x_n^j)$ be the production schedule or vector for facility j. The vector k is given by

$$k = \begin{bmatrix} (x^1, x^2, \ldots, x^N) \end{bmatrix} = \begin{bmatrix} x_1^1, x_1^2, \ldots, x_1^N \\ x_2^1, x_2^2, \ldots, x_2^N \\ \cdot \\ \cdot \\ x_n^1, x_n^2, \ldots, x_n^N \end{bmatrix}$$

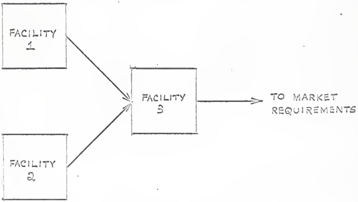
where k is the schedule for the entire network. Often it is necessary to consider the production in facilities j through N.

Let $k^j = (x^j, x^{j+1}, \ldots, x^N)$ be defined as a partial production vector. The vectors $y^j = (y_1^j, y_2^j, \ldots, y_n^j)$ and $r^j = (r_1^j, r_2^j, \ldots, r_n^j)$ represent



ARROWS DENOTE FLOW MATERIALS

Fig 3a. A MULTIPRODUCT SYSTEM



ARROWS DENOTE FLOW OF MATERIALS

Fig 3c. A MULTIPRODUCT SUBASSEMBLY SYSTEM

EACH BLOCK DENOTES A FACILITY. THE LINES INDICATE FLOW OF MATERIAL

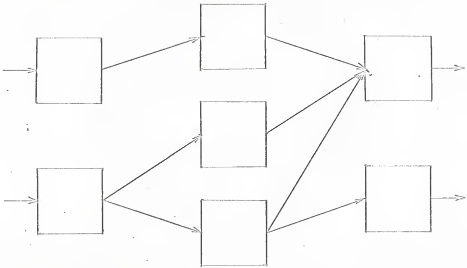


Fig 4. A COMPLEX MULTIPRODUCT MULTIFACILITY SYSTEM

respectively the total demand and market requirements for facility j.

2.5 THE COST STRUCTURE OF THE ACYCLIC NETWORK

For the N stage network the total cost $F(k)$ is given by

$$F(k) = P(k) + \sum_{j=1}^{j=N} \sum_{i=1}^{i=n} M_i^j(I_i^j) \tag{4}$$

where

$P(k)$ = concave cost function of production schedule vector k,

$\sum_{j=1}^{j=N} \sum_{i=1}^{i=n} M_i^j\,(I_i^j)$ = sum of the inventory costs.

Here $P(k)$ represents the joint costs among facilities and can include production and set up costs that are concave functions of k. Each $M_i^j\,(I_i^j)$ is concave on the interval $(-\infty, 0)$ and on the interval $(0, +\infty)$ but need not be concave on the interval $(-\infty, +\infty)$. A function of this form is called piecewise concave. An example of such a function is given in Fig. 5.

Since I_i^j is actually a function of production vector k, it is often convenient to denote this relationship explicitly as $I_i(k)$, so that $I_i^j = I_i^j\,(k)$.

The inventory cost functions can be expressed in terms of k. Let

$$M_i^j\,(k) = M_i^j\,\{I_i^j(k)\}\,.$$

Total cost function is thus

$$F(k) = P(k) + \sum_{j=1}^{j=N} \sum_{i=1}^{i=n} M_i^j(k)\,.$$

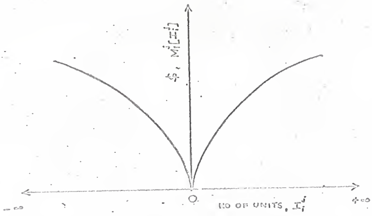


Fig 5. THE INVENTORY COST FUNCTION

$F(k)$ is also piecewise concave.

2.6 PROBLEM STATEMENT

The problem can be stated as follows:

Given certain fixed non negative market requirements for each of the N facilities over the next n periods, find an optimal production schedule k, which minimizes the piecewise concave function

$$F(k) = P(k) + \sum_{j=1}^{j=N} \sum_{i=1}^{i=n} M_i^j(k) \tag{4a}$$

subject to

$$I_i^j = \sum_{h=1}^{h=i} (x_h^j - y_h^j), \tag{5}$$

y_i^j = total demand in period i for facility j

$$= r_i^j + \sum_{h=j+1}^{h=N} a^{jh}\, x_{i+L_h}^h \ , \tag{6}$$

$$I_i^j \geq - \sum_{h=i-\alpha_j+1}^{h=i} y_h^j \ , \tag{7}$$

where the minus sign indicates that negative inventory or backlog is permitted,

$$I_n^j = 0, \tag{8}$$

$$x_i^j \geq 0, \tag{9}$$

for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, N$.

Let x be the set of all production vectors that satisfy the equations (5), (6), (7), (8), and (9); x being a bounded polyhedral set is convex and compact. Any k in x is called feasible. A partial production vector k^h is said to be feasible if equations (5) through (9) hold for $j \geq h$ and all i.

If k^{h+1} is feasible, x^h is said to supply k^{h+1} feasibly if the production vector $k^h = (x^h, k^{h+1})$ is feasible.

Optimization problems have been solved for mainly two cases.

1) Parallel Facility Case,

2) Series Facility Case.

2.7 PARALLEL FACILITY CASE

Here there are N different facilities, producing N different products for which the demand is known exactly for n periods. The problem is to plan an optimal production schedule as to how much each facility should produce in each period in order to minimize total costs. On the surface this might look like a direct generalization of production schedule of N single facilities. But one important difference here is that the fixed cost has been considered as a joint cost for all the N facilities.

Production and inventory costs may vary for each facility. The parallel facility case looks as shown in Fig. 6.

2.8 SERIES FACILITY CASE

In series case, there are N facilities connected in series and only one final product is produced to supply the market requirement.

Product produced in facility j goes to facility (j+1). The final product comes out of facility N. In order that facility (j+1) produces one unit, it may be necessary that facility j should produce 1, 2, or 3 or even more products. But in any case, each facility in the link should produce at least one product. The inventory costs may vary from facility to facility, but the production cost may be expressed as a total cost for all the facilities in a given period.

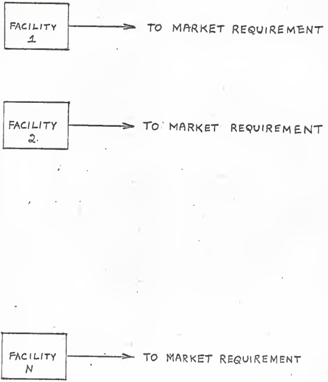


Fig. 6. THE PARALLEL FACILITY CASE

3. CASE STUDY OF PARALLEL FACILITY

3.1 A DYNAMIC PROGRAMMING ALGORITHM FOR PARALLEL FACILITY CASE

As described earlier, a parallel facility case consists of N facilities, each of which supplies only market requirements and no other facilities, so that $a^{jh} = 0$ for all j and h.

The form of the parallel facility case is depicted in Fig. 6. In the parallel facility case the inventory equation (5) reduces to:

$$I_i^j = \sum_{h=1}^{h=i} (x_h^j - r_h^j), \qquad i = 1, 2, \ldots, n,$$
$$j = 1, 2, \ldots, N.$$

One of the interesting aspects of the parallel model is its cost structure. It is assumed that there are joint costs among facilities in each period so that the cost in period i depends upon the production completed and inventories in all N facilities in period i. This assumption permits, for example, inclusion of a cost on the total production completed in all facilities in period i,

$$\sum_{h=1}^{h=N} x_i^h .$$

To express these costs mathematically, let

$$x_i = (x_i^1, x_i^2, x_i^3, \ldots, x_i^N)$$

and

$$I_i = (I_i^1, I_i^2, \ldots, I_i^N),$$

where x_i is a vector representing the production completed in all N facilities in period i, while I_i is a vector representing the inventories in all N facilities in period i. The concave cost term P(k) is expressed as:

$$P(k) = \sum_{i=1}^{n} P_i^1(x_i, I_i)$$

where $P_i^1(x_i, I_i)$ is the cost in period i of having the entire network complete x_i and finish the period with inventories I_i.

To formulate the algorithm consider first the inventory structure of the model. Let us define D as a dominant set which represents all the feasible production vectors. Following a schedule in D the inventory in facility j in period i can be specified by an integer u^j as follows

$$I_i^j = \sum_{h=i+1}^{h=u^j} r_h^j . \qquad (10)$$

The integer u^j indicates that in period i facility j has a stock on hand to satisfy requirements through period u^j. When the j facility's inventory level in period i can be expressed as in equation (10), the inventory horizon is said to be u^j.

Let $u = (u^1, u^2, \ldots, u^N)$ be the vector of u^j. The inventory horizon in the entire network in period i, is said to be u if facility j's inventory horizon in the period i is u^j for all j. Let

$$U_i = \{u | n \geq u_j \geq \max(0, i - \alpha_j)\} .$$

For any schedule in D the network inventory horizon in period i must be u for some u in U_i to maintain feasibility.

The production vectors x_i have an analogous characterization for a schedule in D. If the level of I_{i-1} is u, then

$$x_i^j = \sum_{h=1+u^j}^{h=v^j} r_h^j$$

for some integer v^j where

$$v^j \geq u^j .$$

Let us define a vector

$$v = (v^1, v^2, \ldots, v^N)$$

and a set

$$V_i(u) = \left\{ v \mid \begin{array}{l} \text{if } u^j \geq i \text{ then } v^j = u^j \\ \\ \text{if } u^j < i \text{ then } n \geq v^j \geq \max \ (u^j, i - \alpha) \end{array} \right\}.$$

The network production completed in period i, x_i, is said to supply u+1 to v if $x_i^j = \sum_{h=1+u^j}^{h=v^j} r_h^j$ for all j.

For a schedule in D if the level of I_{i-1} is u and x_i supplies from period u+1 to v then v must be in $V_i(u)$. Furthermore the horizon of I_i must be v.

It is also necessary to consider the cost functions. Let

$$P_i^*(u,v) = P_i^1(x_i, I_i) + \sum_{j=1}^{N} M_i^j (I_i^j) , \tag{11}$$

if the horizon of I_{i-1} is u and x_i supplies from period u+1 to v. $P_i^*(u, v)$ is then the total cost for the network in period i.

The dynamic programming (1) recursion relationship can now be formulated. Let $F_i(u)$ be the minimum cost in all N facilities from period i to n following an optimal production policy given that the horizon of I_{i-1} is u. The recursion is

$$F_i(u) = \min v \in V_i(u) \ \{P_i^*(u, v) + F_{i+1}(v)\} \tag{12}$$

where u is in the set U_{i-1} .

The recursion states that following a schedule in dominant set D at the end of period i-1, the network inventory level will be u for some u in U_{i-1}. The network production completed will then supply from u+1 to v where v is in $V_i(u)$. $P_i^*(u, v)$ is the total network production and inventory charges in period i. Since at the end of period i, the inventory horizon is v, the cost from the beginning of period i+1 to the end of period n is $F_{i+1}(u)$, i = 1, 2, ..., n-1.

Equation (12) can be used recursively until we get $F_1(u)$ which gives the total cost of all the N facilities in n periods.

Let us illustrate the parallel facility case by a numerical example.

3.2 EXAMPLE 1. PARALLEL FACILITY WITH LINEAR COST FUNCTION

Consider the following 3 facility 3 period situation. Let the demand of the product of each facility be as follows:

	DEMAND IN PERIOD		
FACILITY	1	2	3
1	2	3	4 units
2	1	2	1 units
3	3	1	4 units

No production lag or backlog is permitted in any facility. The production costs for each facility are as follows:

FACILITY	PRODUCTION COST
1	\$ 4/ unit
2	\$ 4/ unit
3	\$ [illegible]/ unit

The holding cost in each of the three facilities is \$2/ unit/period. The joint fixed cost for all the three facilities is

$$\delta_i\ (0) = 0, \qquad i = 1, 2, 3,$$

$$\delta_1\ (x) = 2, \qquad x > 0,$$

$$\delta_2\ (x) = 3, \qquad x > 0,$$

$$\delta_3\ (x) = 2, \qquad x > 0,$$

where x is any production level. This means that the fixed cost in any given period is zero only if none of the three facilities produces. If any one facility produces, the fixed cost is \$2 in period 1, \$3 in period 2 and \$2 in period 4. Plan an optimal production schedule to minimize the overall cost.

3.3 SOLUTION BY THE DYNAMIC PROGRAMMING

By looking at the problem we note that production cost in each period depends upon the sum of the productions in each facility and is characterized by a set up cost plus a linear cost term. The inventory cost is a function of the sum of the three inventories and is linear.

To determine the optimal schedule note that the linear portion of the production cost can be neglected since each facility by the end of period 3 must satisfy all of its market requirements.

The equations employed to calculate the cost are, as derived before

$$P_i^*\ (u, v) = P_i^*\ \{(u^1, u^2, u^3), (v^1, v^2, v^3)\}$$

$$= \delta_i\left(\sum_{h=u^1+1}^{h=v^1} r_h^1 + \sum_{h=u^2+1}^{h=v^2} r_h^2 + \sum_{h=u^3+1}^{h=v^3} r_h^3 \right)$$

$$+ 4\left(\sum_{h=u^1+1}^{h=v^1} r_h^1 + \sum_{h=u^2+1}^{h=v^2} r_h^2 + \sum_{h=u^3+1}^{h=v^3} r_h^3 \right)$$

$$+ 2 \left(\sum_{h=i+1}^{v^1} r_h^1 + \sum_{h=i+1}^{v^2} r_h^2 + \sum_{h=i+1}^{v^3} r_h^3 \right),$$

$$i = 1, 2, 3.$$

As mentioned earlier, to simplify the calculations, we shall neglect the linear portion of the production cost which is

$$4 \left(\sum_{h=u^1+1}^{v^1} r_h^1 + \sum_{h=u^2+1}^{v^2} r_h^2 + \sum_{h=u^3+1}^{v^3} r_h^3 \right).$$

The other equation employed recursively is

$$F_i(u) = \min v \text{ in } V_i(u) \; \{P_i^*(u, v) + F_{i+1}(v)\}$$

and obviously

$$F_3(u) = P_3^*(u, v) .$$

The calculations are shown below. When there is more than one possible decision at any step of the calculation, the optimal decision is listed:

$$F_3(3, 3, 3) = P_3^* \; \{ (3, 3, 3), (3, 3, 3) \} = \$0 .$$

Note that $P_3^* \{ (3, 3, 3), (3, 3, 3) \}$ is the cost of period 3. The u^j are $(3, 3, 3)$ for $j = 1, 2, 3$ which means that in the beginning of period 3, the inventory horizons of all the three facilities are $(3, 3, 3)$, with the result that no facility need produce anything and hence the cost is zero. Now consider

$$F_3(2, 3, 3) = P_3 \; \{ (2, 3, 3), (3, 3, 3) \} .$$

Here $u^1 = 2$, $u^2 = 3$, $u^3 = 3$, $v^1 = 3$, $v^2 = 3$, $v^3 = 3$.

This means that in the beginning of period 3, the inventory horizon of facility 1 (u^1=2) is such that it can satisfy the demand of up to and including period 2. Obviously it must produce in the third period in order to satisfy the demand of third period hence giving rise to production cost. Since $u^2 = 3$ and $u^3 = 3$, the inventory horizons of facilities 2 and 3 are enough to satisfy their respective demands in period 3. Hence, $F_3(2,3,3)$=\$2. This is the set up portion of the production cost since $d_3(x)=2$ for $x > 0$.

Similarly we can calculate the relevant costs for other production vectors.

$$F_3(2, 2, 3) = \$2,$$

$$F_3(2, 3, 2) = \$2,$$

$$F_3(3, 3, 2) = \$2,$$

$$F_3(3, 2, 3) = \$2,$$

$$F_3(3, 2, 2) = \$2,$$

$$F_3(2, 2, 2) = \$2 .$$

This completes the calculation of the cost for the third period. Now we will go on to calculate the total cost for third and second periods.

From equation (12) we have

$$F_2(u) = \min v \text{ in } V_2(u) \{P_1^*(u, v) + F_3(v)\} ,$$

$$F_2(3, 3, 3) = P_2^* \{ (3, 3, 3), (3, 3, 3) \} + F_3(3, 3, 3)$$

$$= \$18.$$

Here $u^1 = 3$, $u^2 = 3$, $u^3 = 3$. This shows that inventory levels of all the three facilities are enough to satisfy the demand up to and including the

third period. Hence there is no production. The only cost incurred is the holding cost of the demand for the third period. In the third period, facility 1 has a demand of 4 units, facility 2 has a demand of 1 unit and facility 3 has a demand of 4 units. Hence the holding cost = $\$(4 + 1 + 4) \times 2$ = \$18, that is

$$F_2(3, 3, 3) = \$18.$$

$$F_2(2, 3, 3) = \text{min of} \left\{ \begin{array}{c} P_2^* \ \{(2, 3, 3), (2, 3, 3)\} + F_3(2, 3, 3) \\ \text{or} \\ P_2^* \ \{(2, 3, 3), (3, 3, 3)\} + F_3(3, 3, 3) \end{array} \right\},$$

$$P_2^* \ \{(2, 3, 3)\ (2, 3, 3)\} = (1 + 4)\ 2 = \$10, \text{ only inventory cost,}$$

$$P_2^* \ \{(2, 3, 3)\ (2, 3, 3)\} + F_3(2, 3, 3) = \$10 + 2 = \$12 .$$

Similarly

$$P_2^* \ \{(2, 3, 3)\ (3, 3, 3)\} + F_3(3, 3, 3) = (4 + 1 + 4)\ 2 + 3 + 0 = \$21.$$

Hence the first one of the two plans should be adopted.

$$F_2(2, 3, 3) = P_2^* \ \{(2, 3, 3)\ (2, 3, 3)\} + F_3(2, 3, 3)$$

$$= \$12 \text{ decision } (2, 3, 3) .$$

$F_2(2, 2, 3)$ = Minimum of the following:

$$P_2^* \ \{(2, 2, 3)\ (2, 2, 3)\} + F_3\ (2, 2, 3) \text{ or}$$

$$P_2^* \ \{(2, 2, 3)\ (2, 3, 3)\} + F_3\ (2, 3, 3) \text{ or}$$

$$P_2^* \ \{(2, 2, 3)\ (3, 3, 3)\} + F_3\ (3, 3, 3)$$

$$= \$10 \text{ decision } (2, 2, 3) .$$

Similarly

$F_2\ (3, 3, 2) = \$12$ decision $(3, 3, 2)$,

$F_2\ (3, 2, 3) = \$18$ decision $(3, 2, 3)$,

$F_2\ (3, 2, 2) = \$10$ decision $(3, 2, 2)$,

$F_2\ (2, 2, 2) = \$2$ decision $(2, 2, 2)$,

$F_2\ (2, 3, 2) = \$4$ decision $(2, 3, 2)$.

$F_2\ (3, 3, 1) =$ Minimum of the following:

$$\left\{ P_2^* \left\{ (3, 3, 1), (3, 3, 2) \right\} + F_3\ (3, 3, 2), \right.$$

$$\left. P_2^* \left\{ (3, 3, 1), (3, 3, 3) \right\} + F_3\ (3, 3, 3) \right\}$$

$= \$13$ decision $(3, 3, 2)$,

$F_2\ (3, 2, 1) = \$11$ decision $(3, 2, 2)$,

$F_2\ (3, 1, 1) = \$11$ decision $(3, 2, 2)$,

$F_2\ (2, 3, 1) = \$5$ decision $(2, 3, 2)$,

$F_2\ (2, 2, 1) = \$5$ decision $(2, 2, 2)$,

$F_2\ (2, 1, 1) = \$5$ decision $(2, 2, 2)$,

$F_2\ (1, 3, 1) = \$5$ decision $(2, 3, 2)$,

$F_2\ (1, 2, 1) = \$5$ decision $(2, 2, 2)$,

$F_2\ (1, 1, 1) = \$3$ decision $(2, 2, 2)$,

$F_2\ (3, 1, 2) = \$11$ decision (3, 2, 2),

$F_2\ (2, 1, 2) = \$5$ decision (2, 2, 2),

$F_2\ (1, 3, 2) = \$5$ decision (2, 3, 2),

$F_2\ (1, 2, 2) = \$5$ decision (2, 2, 2),

$F_2\ (1, 1, 2) = \$5$ decision (2, 2, 2) .

This completes the calculation of the total costs for the second and third periods. We will now calculate total costs for the first, second and third periods. In this case $u^1 = 0$, $u^2 = 0$, $u^3 = 0$. Since there is no production before period 1, therefore,

$F_1\ (0, 0, 0)$ = minimum of the following:

$P_1^*\ \{(0, 0, 0)\ (1, 1, 1)\} + F_2\ (1, 1, 1)$,

$P_1^*\ \{(0, 0, 0)\ (1, 1, 2)\} + F_2\ (1, 1, 2)$,

$P_1^*\ \{(0, 0, 0)\ (1, 1, 3)\} + F_2\ (1, 1, 3)$,

$P_1^*\ \{(0, 0, 0)\ (1, 2, 1)\} + F_2\ (1, 2, 1)$,

$P_1^*\ \{(0, 0, 0)\ (1, 2, 2)\} + F_2\ (1, 2, 2)$,

$P_1^*\ \{(0, 0, 0)\ (1, 2, 3)\} + F_2\ (1, 2, 3)$,

$P_1^*\ \{(0, 0, 0)\ (1, 3, 1)\} + F_2\ (1, 3, 1)$,

$P_1^*\ \{(0, 0, 0)\ (1, 3, 2)\} + F_2\ (1, 3, 2)$,

$P_1^*\ \{(0, 0, 0)\ (1, 3, 3)\} + F_2\ (1, 3, 3)$,

$P_1^*\ \{(0, 0, 0)\ (2, 1, 1)\} + F_2\ (2, 1, 1)$,

$P_1^*\ \{(0, 0, 0)\ (2, 1, 2)\} + F_2\ (2, 1, 2)$,

$$P_1^* \{(0, 0, 0)\ (2, 1, 3)\} + F_2\ (2, 1, 3),$$

$$P_1^* \{(0, 0, 0)\ (2, 2, 1)\} + F_2\ (2, 2, 1),$$

$$P_1^* \{(0, 0, 0)\ (2, 2, 2)\} + F_2\ (2, 2, 2),$$

$$P_1^* \{(0, 0, 0)\ (2, 2, 3)\} + F_2\ (2, 2, 3),$$

$$P_1^* \{(0, 0, 0)\ (2, 3, 1)\} + F_2\ (2, 3, 1),$$

$$P_1^* \{(0, 0, 0)\ (2, 3, 2)\} + F_2\ (2, 3, 2),$$

$$P_1^* \{(0, 0, 0)\ (2, 3, 3)\} + F_2\ (2, 3, 3),$$

$$P_1^* \{(0, 0, 0)\ (3, 1, 1)\} + F_2\ (3, 1, 1),$$

$$P_1^* \{(0, 0, 0)\ (3, 1, 2)\} + F_2\ (3, 1, 2),$$

$$P_1^* \{(0, 0, 0)\ (3, 1, 3)\} + F_2\ (3, 1, 3),$$

$$P_1^* \{(0, 0, 0)\ (3, 2, 1)\} + F_2\ (3, 2, 1),$$

$$P_1^* \{(0, 0, 0)\ (3, 2, 2)\} + F_2\ (3, 2, 2),$$

$$P_1^* \{(0, 0, 0)\ (3, 2, 3)\} + F_2\ (3, 2, 3),$$

$$P_1^* \{(0, 0, 0)\ (3, 3, 3)\} + F_2\ (3, 3, 3),$$

$$= \$7 \quad \text{decision } (1, 1, 1)\ .$$

This means that in the first period the optimal schedule is to produce as much as needed for the first period only. Corresponding to this optimal schedule for the second period is (2, 2, 2) which means to produce as much as needed for the second period only.

Hence the optimal schedule is as follows:

	Production level in period		
FACILITY	1	2	3
1	2	3	4 units
2	1	2	1 units
3	3	1	4 units

Now we have to calculate the total cost corresponding to the optimal schedule.

F_1 (0, 0, 0) = \$7 does not include the linear part of the production costs.

Total cost = \$7 + Production cost x Total units

= \$7 + 21 x 4 = \$91.

3.4 SOLUTION BY THE DISCRETE MAXIMUM PRINCIPLE

The discrete maximum principle (2, 5) is a powerful tool in handling multistage optimization problems. The above problem which has been solved by dynamic programming can be solved in more elegant and efficient manner with the discrete maximum principle.

Let us define the 3 periods as 3 stages as represented in Fig. 7. Thus n = 1, 2, 3 and N represents the final stage. Hence N = 3.

Let

x_1^n = Production level of the first facility in the n^{th} period,

x_2^n = Production level of the second facility in the n^{th} period,

x_3^n = Production level of the third facility in the n^{th} period,

θ_1^n = Change in production level from $(n-1)^{th}$ period (stage) to n^{th} period at the first facility,

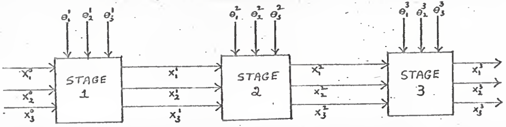


Fig 7 REPRESENTATION OF PROCESS STREAM FOR PARALLEL FACILITY CASE

θ_2^n = Change in production level from $(n-1)^{th}$ stage to n^{th} stage at the second facility,

θ_3^n = Change in production level from $(n-1)^{th}$ stage to n^{th} stage at the third facility,

Q_i^n = Sales forecast for the product of i^{th} facility in the n^{th} period,

$$i = 1, 2, 3 .$$

The transformation of the process stream at the n^{th} stage, described by a set of performance equations, is as follows

$$x_1^n = T(x_1^{n-1}, \theta_1^n) = x_1^{n-1} + \theta_1^n, \quad n = 1, 2, 3, \tag{13}$$

$$x_1^0 = 0 \text{ and } x_1^n \geq Q_1^n, \tag{14}$$

$$x_2^n = T(x_2^{n-1}, \theta_2^n) = x_2^{n-1} + \theta_2^n, \quad n = 1, 2, 3, \tag{15}$$

$$x_2^0 = 0 \text{ and } x_2^n \geq Q_2^n, \tag{16}$$

$$x_3^n = T(x_3^{n-1}, \theta_3^n) = x_3^{n-1} + \theta_3^n, \quad n = 1, 2, 3, \tag{17}$$

$$x_3^0 = 0 \text{ and } x_3^n \geq Q_3^n, \tag{18}$$

$$F^n = \text{Fixed cost at the } n^{th} \text{ period .} \tag{19}$$

Let us introduce a new state variable x_4^n to represent cost such as

x_4^n = The sum of the costs up to and including the n^{th} stage (period),

$G(x^{n-1}, \theta^n)$ = The cost at the n^{th} stage.

Thus

$$
\begin{aligned}
x_4^n &= x_4^{n-1} + G(x^{n-1}, \theta^n) \\
&= x_4^{n-1} + 4(x_1^n + x_2^n + x_3^n) \\
&+ 2(I_1^{n-1} + I_2^{n-1} + I_3^{n-1} + x_1^n + x_2^n + x_3^n \\
&- Q_1^n - Q_2^n - Q_3^n) + F^n \qquad (20) \\
&= x_4^{n-1} + 6(x_1^n + x_2^n + x_3^n) + 2(I_1^{n-1} + I_2^{n-1} + I_3^{n-1}) \\
&- 2(Q_1^n + Q_2^n + Q_3^n) + F^n , \qquad (20a)
\end{aligned}
$$

$$x_4^0 = 0 . \qquad (20b)$$

where I_i^n is the inventory level of the i^{th} facility in the n^{th} period.

We shall define the cost function in terms of state variables of the final stage. Thus

$$S = \sum_{i=1}^{i=4} c_i x_i^N = x_4^N . \qquad (21)$$

Hence

$$c_i = 0, \quad i = 1, 2, 3,$$

$$c_4 = 1 .$$

The function S which is to be minimized is the objective function of the process.

The procedure for solving such an optimization problem by the discrete maximum principle is to introduce a four dimensional adjoint vector z^n and a Hamiltonian function H^n which satisfies the follow· · relationship

$$H^n = \sum_{i=1}^{4} z_i^n x_i^n .$$

Substituting for x_1^n, x_2^n, x_3^n and x_4^n from equation (13), (15), (17) and (20a) respectively, we get

$$\begin{aligned} H^n = {} & z_1^n(x_1^{n-1} + \theta_1^n) + z_2^n(x_2^{n-1} + \theta_2^n) + z_3^n(x_3^{n-1} + \theta_3^n) \\ & + z_4^n \{ x_4^{n-1} + 4(x_1^n + x_2^n + x_3^n) + \\ & + 2(I_1^{n-1} + I_2^{n-1} + I_3^{n-1} + x_1^n + x_2^n + x_3^n \\ & - Q_1^n - Q_2^n - Q_3^n) + F^n \} . \end{aligned}$$

Rearranging the terms, we obtain

$$\begin{aligned} H^n = {} & z_1^n x_1^{n-1} + z_2^n x_2^{n-1} + z_3^n x_3^{n-1} + z_1^n \theta_1^n + z_2^n \theta_2^n + z_3^n \theta_3^n \\ & + z_4^n \{ x_4^{n-1} + 6(x_1^{n-1} + x_2^{n-1} + x_3^{n-1}) + 6(\theta_1^n + \theta_2^n + \theta_3^n) \\ & + 2(I_1^{n-1} + I_2^{n-1} + I_3^{n-1}) - 2(Q_1^n + Q_2^n + Q_3^n) + F^n \} . \end{aligned} \tag{22}$$

The adjoint variable z_i^{n-1} is defined as

$$z_i^{n-1} = \frac{\partial H^n}{\partial x_i^{n-1}} , \quad i = 1, 2, 3, 4, \quad n = 1, 2, 3, \tag{23}$$

and

$$z_i^3 = c_i , \qquad i = 1, 2, 3, 4 . \tag{24}$$

Hence

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = z_1^n + 6z_4^n , \tag{25}$$

$$z_2^{n-1} = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n + 6z_4^n , \tag{26}$$

$$z_3^{n-1} = \frac{\partial H^n}{\partial x_3^{n-1}} = z_3^n + 6z_4^n , \tag{27}$$

$$z_4^{n-1} = \frac{\partial H^n}{\partial x_4^{n-1}} = z_4^n \quad . \tag{28}$$

But from equations (24) and (28), we have

$$z_4^3 = c_4 = 1,$$

and

$$z_4^{n-1} = z_4^n = 1, \quad n = 1, 2, 3 .$$

Substituting this in equation (22) yields

$$\begin{aligned} H^n = {} & z_1^n x_1^{n-1} + z_2^n x_2^{n-1} + z_3^n x_3^{n-1} + z_1^n \theta_1^n + z_2^n \theta_2^n \\ & + z_3^n \theta_3^n + x_4^{n-1} + 6(x_1^{n-1} + x_2^{n-1} + x_3^{n-1}) + 6(\theta_1^n + \theta_2^n + \theta_3^n) + \\ & + 2(I_1^{n-1} + I_2^{n-1} + I_3^{n-1}) - 2(Q_1^n + Q_2^n + Q_3^n) + F^n . \end{aligned} \tag{29}$$

Noting that in equation (29) x_i^{n-1}, I_i^{n-1}, Q_i^n and z_i^n for $i = 1, 2, 3, 4$ are considered to be constants, H^n can be written as

$$H^n = H^n_c + H^n_v$$

where H^n_c is the constant part of H^n, and H^n_v is the variable part of H^n. H^n_v is given as

$$H^n_v = \theta^n_1(z^n_1 + 6) + \theta^n_2(z^n_2 + 6) + \theta^n_3(z^n_3 + 6) + F^n . \qquad (30)$$

The objective function, S, of equation (21) is a minimum where the Hamiltonian function H^n is a minimum. Since the performance equations (14), (15), and (16) are linear in their arguments,

$$H^n = \text{Minimum},$$

is a necessary as well as sufficient condition for objective function S to be a minimum (4). Obviously H^n is a minimum when H^n_v, which is linear in θ^n_i, is a minimum for n = 1, 2, 3. The optimal value of the decision variable $\bar{\theta}_i$ is that value of θ_i which makes H^n_v a minimum.

Now let us evaluate the values of z^n_i at each stage. From equation (24)

$$z^3_i = c_i = 0, \quad i = 1, 2, 3. \qquad (31)$$

and from equations (25), (26), (27), and (28), we find,

$$z^{n-1}_i = z^n_i + 6z^n_4$$

$$= z^n_i + 6, \text{ since } z^n_4 = 1, \quad i = 1, 2, 3.$$

Hence

$$z^2_i = z^3_i + 6$$

$$= 0 + 6 = 6, \quad i = 1, 2, 3, \qquad (32)$$

$$z_i^1 = z_i^2 + 6$$

$$= 6 + 6 = 12, \quad i = 1, 2, 3 \; . \tag{33}$$

Now we will find those values of θ_i which will make H_v^n a minimum for $n = 1, 2, 3$.

Stage 1.

Substituting the values of z_i^1 given by equation (33) into equation (30), we obtain

$$H_v^1 = \theta_1^1 \, (12 + 6) + \theta_2^1 \, (12 + 6) + \theta_3^1 \, (12 + 6) + F^n$$

$$= 18 \, (\theta_1^1 + \theta_2^1 + \theta_3^1) + F^n \; . \tag{34}$$

Obviously H_v^1 is a minimum when $(\theta_1^1 + \theta_2^1 + \theta_3^1)$ is a minimum. But from equations (13), (14), (15) and (16)

$$x_i^1 = x_i^0 + \theta_i^1$$

$$= 0 + \theta_i^1$$

$$= \theta_i^1 \; , \qquad i = 1, 2, 3,$$

and

$$x_i^1 \geq Q_i^1 \; , \qquad i = 1, 2, 3 \; .$$

It follows that

$$x_i^1 = \theta_i^1 = Q_i^1 \text{ makes } H_v^n \text{ a minimum} \; .$$

Hence

$$x_1^1 = \theta_1^1 = Q_1^1 = 2,$$

$$x_2^1 = \theta_2^1 = Q_2^1 = 1,$$

$$x_3^1 = \theta_3^1 = Q_3^1 = 3 .$$

Stage 2.

$$H_v^2 = \theta_1^2 (z_1^2 + 6) + \theta_2^2 (z_2^2 + 6) + \theta_3^2 (z_3^2 + 6) + F^2 \quad . \qquad (35)$$

Substituting the values of z_i^2 given by equation (32), we have

$$H_v^2 = \theta_1^2 (6 + 6) + \theta_2^2 (6 + 6) + \theta_3^2 (6 + 6) + F^2$$

$$= 12 (\theta_1^2 + \theta_2^2 + \theta_3^2) + F^2 .$$

This is a minimum when $(\theta_1^2 + \theta_2^2 + \theta_3^2)$ is a minimum subject to the constant $x_i^2 \geq Q_i^2$, i = 1, 2, 3 .

Hence

$$x_1^2 = x_1^1 + \theta_1^2 \geq Q_1^2 ,$$

that is,

$$x_1^2 = 2 + \theta_1^2 \geq 3$$

Minimum $\theta_1^2 = 1$ and $x_1^2 = 3$.

$$x_2^2 = x_2^1 + \theta_2^2 \geq Q_2^2$$

that is

$$x_2^2 = 1 + \theta_2^2 \geq 2$$

Minimum $\theta_2^2 = 1$ and $x_2^2 = 2$.

$$x_3^2 = x_3^1 + \theta_3^2 \geq Q_3^2$$

that is

$$x_3^2 = 3 + \theta_3^2 \geq 1$$

Minimum $\theta_3^2 = -2$ and $x_3^2 = 1$

Stage 3.

$$H_v^3 = \theta_1^3 \, (z_1^3 + 6) + \theta_2^3 \, (z_2^3 + 6) + \theta_3^3 \, (z_3^3 + 6) + F^2 \; .$$

Substituting $z_i^3 = 0$ into the equation yields

$$H_v^3 = 6(\theta_1^3 + \theta_2^3 + \theta_3^3) + F^3 \; .$$

H_v^3 is a minimum when $(\theta_1^3 + \theta_2^3 + \theta_3^3)$ is a minimum, subject to satisfy the constraint $x_i^3 \geq Q_i^3$, i = 1, 2, 3.

That is

$$x_1^3 = x_1^2 + \theta_1^3 \geq Q_1^3$$

$$= 3 + \theta_1^3 \geq 4 \; .$$

Minimum $\theta_1^3 = 1$ and $x_1^3 = 4$.

Minimum $\theta_1^3 = 1$ and $x_1^3 = 4$.

$$x_2^3 = x_2^2 + \theta_2^3 \geq Q_2^3$$

$$= 2 + \theta_2^3 \geq 1 \ .$$

Minimum $\theta_2^3 = -1$ and $x_2^3 = 1$.

$$x_3^3 = x_3^2 + \theta_3^3 \geq Q_3^3$$

$$= 1 + \theta_3^3 \geq 4 \ .$$

Minimum $\theta_3^3 = 3$ and $x_3^3 = 4$.

The optimum production schedule can be tabulated as follows:

	PRODUCTION LEVEL IN PERIOD		
FACILITY	1	2	3
1	2	3	4 units
2	1	2	1 units
3	3	1	4 units

The decision vectors are as follows:

n	θ_1^n	θ_2^n	θ_3^n
1	2	1	3
2	1	1	-2
3	1	-1	3

Now we will calculate the total cost, x_4^3, corresponding to the optimal schedule. We have seen from equations (20a) and (20b) that

$$x_4^0 = 0$$

and

$$x_4^n = x_4^{n-1} + 6(x_1^{n-1} + x_2^{n-1} + x_3^{n-1}) + 6\ (\theta_1^n + \theta_2^n + \theta_3^n)$$

$$+ 2\ (I_1^{n-1} + I_2^{n-1} + I_3^{n-3})$$

$$- 2\ (Q_1^n + Q_2^n + Q_3^n) + F^n\ . \tag{36}$$

But we also note that inventory at each stage is zero, since the production level at each stage does not exceed demand. Hence we can rewrite equation (36) as follows:

$$x_4^n = x_4^{n-1} + 6\ (x_1^{n-1} + x_2^{n-1} + x_3^{n-1}) + 6\ (\theta_1^n + \theta_2^n + \theta_3^n)$$

$$- 2\ (Q_1^n + Q_2^n + Q_3^n) + F^n,$$

$$x_4^1 = x_4^0 + 6\ (x_1^0 + x_2^0 + x_3^0) + 6\ (\theta_1^1 + \theta_2^1 + \theta_3^1) - 2\ (Q_1^1 + Q_2^1 + Q_3^1) + F^1$$

$$= 0 + 6\ (0 + 0 + 0) + 6\ (2 + 1 + 3) - 2\ (2 + 1 + 3) + 2$$

$$= 26, \tag{37}$$

$$x_4^2 = x_4^1 + 6\ (x_1^1 + x_2^1 + x_3^1) + 6\ (\theta_1^2 + \theta_2^2 + \theta_3^2) - 2\ (Q_1^2 + Q_2^2 + Q_3^2) + F^2$$

$$= 26 + 6\ (2 + 1 + 3) + 6\ (1 + 1 - 2) - 2\ (3 + 2 + 1) + 3$$

$$= 53, \tag{38}$$

$$x_4^3 = x_4^2 + 6\ (x_1^2 + x_2^2 + x_3^2) + 6\ (\theta_1^3 + \theta_2^3 + \theta_3^3) - 2\ (Q_1^3 + Q_2^3 + Q_3^3) + F^3$$

$$= 53 + 6\ (3 + 2 + 1) + 6\ (1 - 1 + 3) - 2\ (4 + 1 + 4) + 2$$

$$= \$91\ . \tag{39}$$

Hence minimum cost corresponding to optimal schedule = \$91 .

It may be noted that in the above problem the cost function involved is linear. But problems involving concave cost functions also can be worked out in the same manner. In the above problem, it is found that carrying inventory is not profitable and hence production in each period is just sufficient to satisfy the demand. But this is not true always. In some problems it may be found profitable to carry inventory, depending upon the cost structure.

3.5 COMPARISON OF THE METHODS

Having worked out the parallel facility case with both methods namely dynamic programming and the discrete maximum principle, we are ready to draw some conclusions regarding relative merit of each method. One of the obvious conclusions is that the discrete maximum principle and dynamic programming give the same results. However, the discrete maximum principle appears to be a little more powerful in the type of problem discussed above. Dynamic programming will start the investigation by searching the entire grid of n variables at one stage, store this grid of values and proceed stage by stage, while the discrete maximum principle will start the investigation by computing one optimum path along each stage and then proceed to improve this optimum path based on the values obtained from the preceeding computation. It could also be noted that in the solution of the above problem dynamic programming started from the final stage and went backwards whereas the discrete maximum principle proceeded from the first stage.

3.6 EXAMPLE 2. PARALLEL FACILITY CASE WITH NON-LINEAR COST FUNCTION

Consider the following 4 facility, 4 period situations, each producing a perishable commodity. Let the known market requirements of the products of the four facilities be as follows:

FACILITY	Initial Production Level	Demand in Period 1	2	3	4
1	$18\frac{3}{4}$	10	6	7	6 units
2	16	7	6	$4\frac{1}{2}$	$3\frac{1}{2}$ units
3	$16\frac{1}{2}$	8	6	5	4 units
4	17	12	8	$5\frac{1}{2}$	4 units

The excess production over the sales forecast is wasted at the rate of \$5/unit in each facility. The cost of changing the production level is 2 times the square of the difference between two successive production levels in each facility. Plan an optimal schedule to minimize the cost.

3.7 SOLUTION BY THE DISCRETE MAXIMUM PRINCIPLE

Let us define each period as a stage.

Let

x_i^n = Production level at the n^{th} stage of the i^{th} facility, $i=1,2,3,4$,

θ_i^n = Change in production level of the i^{th} facility from $(n-1)^{th}$ stage to n^{th} stage (period), $i = 1, 2, 3, 4$,

Q_i^n = Demand for the product of the first facility in the n^{th} period, $i = 1, 2, 3, 4$.

We can write the performance equations as follows:

$$x_1^n = T(x_1^{n-1},\ \theta_1^n) = x_1^{n-1} + \theta_1^n\ , \quad x_1^0 = 18\ \tfrac{3}{4}\ , \tag{40}$$

$$x_2^n = T(x_2^{n-1},\ \theta_2^n) = x_2^{n-1} + \theta_2^n\ , \quad x_2^0 = 16\ , \tag{41}$$

$$x_3^n = T(x_3^{n-1},\ \theta_3^n) = x_3^{n-1} + \theta_3^n\ , \quad x_3^0 = 16\ \tfrac{1}{2}\ , \tag{42}$$

$$x_4^n = T(x_4^{n-1},\ \theta_4^n) = x_4^{n-1} + \theta_4^n\ , \quad x_4^0 = 17\ . \tag{43}$$

Let us introduce a new state variable x_5^n to represent cost such as

$$x_5^n = \text{Total cost up to and includling } n^{th} \text{ stage}$$

$$= x_5^{n-1} + G\ (x^{n-1};\theta^n) \tag{44}$$

$$= x_5^{n-1} + 2\sum_{i=1}^{i=4} (\theta_i^n)^2 + 5\sum_{i=1}^{i=4} (x_i^n - Q_i^n)\ , \tag{45}$$

$$x_5^0 = 0\ . \tag{46}$$

Substituting equations (40) to (43) in equation (45) we get:

$$\begin{aligned} x_5^n = x_5^{n-1} &+ 2\ \{\ (\theta_1^n)^2 + (\theta_2^n)^2 + (\theta_3^n)^2 + (\theta_4^n)^2\ \} \\ &+ 5\ \{\ (x_1^{n-1} + \theta_1^n - Q_1^n) + (x_2^{n-1} + \theta_2^n - Q_2^n) \\ &+ (x_3^{n-1} + \theta_3^n - Q_3^n) + (x_4^{n-1} + \theta_4^n - Q_4^n)\ \}\ . \end{aligned} \tag{47}$$

The objective function to be minimized is

$$S = \sum_{i=1}^{5} c_i x_i^4 = x_5^4\ . \tag{48}$$

From this it follows

$$c_i = 0,\ i = 1,2,3,4,\ \text{and}\ c_5 = 1\ . \tag{49}$$

Now the Hamiltonian and adjoint variables are given as

$$H^n = \sum_{i=1}^{5} z_i^n x_i^n$$

$$= \sum_{i=1}^{4} z_i^n\,(x_i^{n-1} + \theta_i^n) + z_5^n\,\{\,x_5^{n-1} + 2(\theta_1^n)^2 + 2(\theta_2^n)^2$$

$$+ (\theta_3^n)^2 + (\theta_4^n)^2 + 5\,(x_1^{n-1} + \theta_1^n - Q_1^n) + 5\,(x_2^{n-1} + \theta_2^n - Q_2^n)$$

$$+ 5\,(x_3^{n-1} + \theta_3^n - Q_3^n) + 5\,(x_5^{n-1} + \theta_4^n - Q_4^n)\,\}\ , \tag{50}$$

$$n = 1,\ 2,\ 3,\ 4,$$

$$z_i^{n-1} = \frac{\partial H^n}{\partial x_i^{n-1}} = z_i^n + 5,\ n = 2,\ 3,\ 4, \tag{51a}$$

$$i = 1,\ 2,\ 3,\ 4,$$

$$z_5^{n-1} = \frac{\partial H^n}{\partial x_5^{n-1}} = z_4^n,\ n = 2,\ 3,\ 4, \tag{51b}$$

$$z_i^4 = c_i\ , \qquad i = 1,\ 2,\ 3,\ 4,\ 5\ .$$

From equation (49) it follows,

$$z_i^4 = c_i = 0,\ i = 1,\ 2,\ 3,\ 4,$$

$$z_5^4 = c_5 = 1\ .$$

Substituting in the recursion equations (51a) and (51b) for $n = 2, 3, 4$, we obtain

$$z_i^3 = 5 \;, \qquad z_i^2 = 10 \;, \qquad z_i^1 = 15 \;, \; i = 1, 2, 3, 4,$$

and $z_5^n = 1$, $n = 1, 2, 3, 4$.

S is minimum when H^n is minimum. Differentiating H^n with respect to θ^n and equating to zero, we get the optimal values of θ_i^n for $i = 1, 2, 3, 4$, and $n = 1, 2, 3, 4$. That is,

$$\frac{\partial H^n}{\partial \theta_i^n} = z_i^n + 4\theta_i^n z_5^n + 5 = 0 \;,$$

or

$$\theta_i^n = - \frac{5+z_i^n}{4z_5^n} = - \frac{5+z_i^n}{4} \;, \; i = 1, 2, 3, 4 \;.$$

Values of θ_i^n are accepted only if admissible, i.e., we accept the values only if $x_i^n \geq Q_i^n$, $i = 1, 2, 3, 4$; $n = 1, 2, 3, 4$.

$$\theta_i^1 = - \frac{5+z_i^1}{4} = - \frac{5+15}{4} = -5,$$

$$\theta_i^2 = - \frac{5+z_i^2}{4} = - \frac{5+10}{4} = -3\,\frac{3}{4} \;,$$

$$\theta_i^3 = - \frac{5+z_i^3}{4} = - \frac{5+5}{4} = -\,2\,\frac{1}{2} \;,$$

$$\theta_i^4 = -\frac{5+z_i^4}{4} = -\frac{5+0}{4} = -1\tfrac{1}{4} \quad ,$$

for $i = 1, 2, 3, 4$.

Now we will determine the values of x_i^n, $i = 1, 2, 3, 4$; $n = 1, 2, 3, 4$ such that

$$x_1^1 = \max \begin{cases} x_1^0 + \theta_1^1 = 18\tfrac{3}{4} - 5 = 13\tfrac{3}{4} \\ Q_1^1 = 10 \end{cases}$$

$$= 13\tfrac{3}{4} \;,$$

$$x_1^2 = \max \begin{cases} x_1^1 + \theta_1^2 = 13\tfrac{3}{4} - 3\tfrac{3}{4} = 10 \\ Q_1^2 = 6 \end{cases}$$

$$= 10,$$

$$x_1^3 = \max \begin{cases} x_1^2 + \theta_1^3 = 10 - 2\tfrac{1}{2} = 7\tfrac{1}{2} \\ Q_1^3 = 7 \end{cases}$$

$$= 7\tfrac{1}{2} \;,$$

$$x_1^4 = \max \begin{cases} x_1^3 + \theta_1^4 = 7\tfrac{1}{2} - 1\tfrac{1}{4} = 6\tfrac{1}{4} \\ Q_1^4 = 6 \end{cases}$$

$$= 6\tfrac{1}{4} \;,$$

$$x_2^1 = \max \begin{cases} x_2^0 + \theta_2^1 = 16 - 5 = 11 \\ \\ Q_2^1 = 7 \end{cases}$$

$$= 11,$$

$$x_2^2 = \max \begin{cases} x_2^1 + \theta_2^2 = 11 - 3\,\frac{3}{4} = 7\,\frac{1}{4} \\ \\ Q_2^2 = 6 \end{cases}$$

$$= 7\,\frac{1}{4}\ ,$$

$$x_2^3 = \max \begin{cases} x_2^2 + \theta_2^3 = 7\,\frac{1}{4} - 2\,\frac{1}{2} = 4\,\frac{3}{4} \\ \\ Q_2^3 = 4\,\frac{1}{4} \end{cases}$$

$$= 4\,\frac{3}{4}$$

$$x_2^4 = \max \begin{cases} x_2^3 + \theta_2^4 = 4\,\frac{3}{4} - 1\,\frac{1}{4} = 3\,\frac{1}{2} \\ \\ Q_2^4 = 3\,\frac{1}{2} \end{cases}$$

$$= 3\,\frac{1}{2}\ ,$$

$$x_3^1 = \max \begin{cases} x_3^0 + \theta_3^1 = 16\,\frac{1}{2} - 5 = 11\,\frac{1}{2} \\ \\ Q_3^1 = 8 \end{cases}$$

$$= 11\,\frac{1}{2}\ ,$$

$$x_3^2 = \max \begin{cases} x_3^1 + \theta_3^2 = 11\,\frac{1}{2} - 3\,\frac{3}{4} = 7\,\frac{3}{4} \\ \\ Q_3^2 = 6 \end{cases}$$

$$= 7\,\frac{3}{4}\ ,$$

$$x_3^3 = \max \begin{cases} x_3^2 + \theta_3^3 = 7\,\frac{3}{4} - 2\,\frac{1}{2} = 5\,\frac{1}{4} \\ \\ Q_2^3 = 5 \end{cases}$$

$$= 5\,\frac{1}{4}\ ,$$

$$x_3^4 = \max \begin{cases} x_3^3 + \theta_3^4 = 5\,\frac{1}{4} - 1\,\frac{1}{4} = 4 \\ \\ Q_3^4 = 4 \end{cases}$$

$$= 4\ ,$$

$$x_4^1 = \max \begin{cases} x_4^0 + \theta_4^1 = 17 - 5 = 12 \\ \\ Q_4^1 = 12 \end{cases}$$

$$= 12,$$

$$x_4^2 = \max \begin{cases} x_4^1 + \theta_4^2 = 12 - 3\,\frac{3}{4} = 8\,\frac{1}{4} \\ \\ Q_4^2 = 8 \end{cases}$$

$$= 8\,\frac{1}{4}\,,$$

$$x_4^3 = \max \begin{cases} x_4^2 + \theta_4^3 = 8\,\frac{1}{4} - 2\,\frac{1}{2} = 5\,\frac{3}{4} \\ \\ Q_4^2 = 5\,\frac{1}{2} \end{cases}$$

$$= 5\,\frac{3}{4}\,,$$

$$x_4^4 = \max \begin{cases} x_4^3 + \theta_4^4 = 5\,\frac{3}{4} - 1\,\frac{1}{4} = 4\,\frac{1}{2} \\ \\ Q_4^4 = 4 \end{cases}$$

$$= 4\,\frac{1}{2}\,.$$

Thus the optimal schedule is as follows:

	PRODUCTION LEVEL IN PERIOD			
FACILITY	1	2	3	4
1	$13\frac{3}{4}$	10	$7\frac{1}{2}$	$6\frac{1}{4}$ units
2	11	$7\frac{1}{4}$	$4\frac{3}{4}$	$3\frac{1}{2}$ units
3	$11\frac{1}{2}$	$7\frac{3}{4}$	$5\frac{1}{4}$	4 units
4	13	$8\frac{1}{4}$	$5\frac{1}{2}$	$4\frac{1}{4}$ units

Now we shall calculate the total cost corresponding to the optimal schedule. From equation (47) total cost up to and including the n^{th} stage is

$$
\begin{aligned}
x_5^n = x_5^{n-1} &+ 2\ \{ (\theta_1^n)^2 + (\theta_2^n)^2 + (\theta_3^n)^2 + (\theta_4^n)^n \} \\
&+ 5\ \{ (x_1^{n-1} + \theta_1^n - Q_1^n) + (x_2^{n-1} + \theta_2^n - Q_2^n) \\
&+ (x_3^{n-1} + \theta_3^n - Q_3^n) + (x_4^{n-1} + \theta_4^n - Q_4^n) \}
\end{aligned}
\tag{52}
$$

$$n = 1, 2, 3, 4,$$

$$x_5^0 = 0 .$$

The total minimum cost given by x_5^4 is \$437.50 .

COMMENT Going through the above procedure, we notice that the calculated value of production level x_i^n is always greater than the demand Q_i^n. In fact, as backlog is not permitted, we assume that $x_i^n \geq Q_i^n$. If this is not the case, this method does not yield the optimal solution. This example can also be worked with the dynamic programming method with a very similar procedure (3, 4).

The above problem was treated as a multi-dimensional problem and solved by the discrete maximum principle. But since cost structure for each facility is independent, as a matter of fact, the production schedule for each facility can be worked out individually. In either case we arrive at the same result.

4. CASE STUDY OF SERIES FACILITY

4.1 A DYNAMIC PROGRAMMING ALGORITHM FOR SERIES FACILITY CASE (8)

A schematical representation of a series facility case is shown in Fig. 8. The series structure of the model means that facility $j (j < N)$ supplies facility $j+1$ only, and does not supply market requirements or any other facility. Let us define $a^{j,j+1}$ as the number of units produced by facility j to supply input to facility $j+1$, so that facility $j+1$ might produce one unit. The following relations hold good for the series facility case

1) $a^{j,j+1} > 0$ for $j < N$,
2) $a^{j,h} = 0$ for $h > j+1$ or $h < j$,
3) r_i^j = Demand for the product of j^{th} facility in the i^{th} period,
   $= 0$ for $j < N$,
4) Only facility N supplies the market.

The series case involves the following two assumptions.

1) The concave cost term $P(k)$ can be expressed as

$$P(k) = \sum_{j=1}^{j=n} p_j \, (x^j)$$

where $p_j(x_j)$ is the n period cost if facility j follows the production schedule $x^j = (x_1^j, x_2^j, \ldots, x_n^j)$.

2) The backlog limit for facility $j < N$ is zero, so that $\alpha_j = 0$ for $j < N$. Facility N, however, can backlog.

The inventory equations can be written as

Fig 8. THE SERIES FACILITY CASE

$$I_i^N = \sum_{h=1}^{h=i} (x_h^N - r_h^N)$$

and

$$I_i^j = \sum_{h=1}^{h=i} (x_h^j - a^{j,j+1}\, x_{h+L_{j+1}}^{j+1}) \text{ for } j < N$$

where

L_j = Production lag period of j^{th} facility.

The following two transformations will simplify the inventory equations. First let

$$\bar{x}_i^j = \frac{x_i^j}{a^j} ,$$

$$\bar{I}_i^j = \frac{I_i^j}{a^j} \qquad \text{for all i and j}$$

where

$$a^j = \prod_{h=j}^{h=N-1} a^{h,h+1} \qquad \text{for } j < N$$

and

$$a^N = 1 .$$

This transformation redefines the units in each facility. The second transformation eliminates the effect of production lags.

Let

$$\hat{x}_i^j = \bar{x}_{i+A_j}^j$$

where

$$A_j = \sum_{h=2}^{h=j} L_h \ ,$$

$$\hat{I}_i^j = \bar{I}_{i+A_j}^j \ ,$$

and

$$\hat{r}_i^N = r_{i+A_N}^N \ .$$

Because of the production lags, to insure that there is no production before a facility can receive initial input it is required that $x_h^j = 0$ for $h \leq A_j$ and $r_h^N = 0$ for $h \leq A_N$. Then the inventory equations can be written as

$$\hat{I}_i^N = \sum_{h=1}^{h=i} (\hat{x}_h^N - \hat{r}_h^N) \ , \tag{53}$$

$$\hat{I}_i^j = \sum_{h=1}^{h=i} (\hat{x}_h^j - \hat{x}_h^{j+1}) \text{ for } j < N \ . \tag{54}$$

The above considerations permit the facility productions

$$\hat{x}^j = (\hat{x}_1^j, \hat{x}_2^j, \ldots, \hat{x}_n^j)$$

in a dominant schedule to be specified. First consider facility N and a dominant partial production vector,

$$\hat{k}^N = \hat{x}^N$$

Since $\hat{x}^N$ must satisfy exact requirements there must exist a vector of integers

$$w = (w_0, w_1, w_2, \ldots, w_n)$$

such that

$$0 = w_0 \leq w_1 \leq w_2 \leq \ldots \leq w_n = n$$

and

$$\hat{x}_i^N = \sum_{h=1+w_{i-1}}^{h=w_i} \hat{r}_h^N, \quad i = 1, 2, \ldots, N .$$

Furthermore since facility N has a backlog limit α_N, to prevent excess backlogging and satisfy equation

$$I_i^j \geq - \sum_{h=i-\alpha_j+1}^{h=i} y_h^j , \tag{55}$$

it is sufficient that

$$w_i \geq i - \alpha_N \quad . \tag{56}$$

Let

$$W = (w \,|\, 0 = w_0 \leq w_1 \leq \ldots \leq w_n = n) .$$

Given any $\hat{x}^N$ in the partial dominant set D^N, there is a w in W such that

$$\hat{x}_i^N = \sum_{h=1+w_{i-1}}^{h=w_i} r_h^N , \quad i = 1, 2, \ldots, n .$$

It is obvious that the production level of facility N in the i^{th} period is the demand of facility N in the same period. This can be represented as follows

$$\hat{y}_i^{N-1} = \hat{x}_i^N$$

$$= \sum_{h=1+w_{i-1}}^{h=w_i} \hat{r}_h^N$$

for some w in W.

Assume now that a vector w^1 in W is given. Let facility N-1 face demand $\hat{y}^{N-1}$. Let us express the demand in the i^{th} period for facility N-1 as follows

$$\hat{y}_i^{N-1} = \sum_{h=1+w_{i-1}^1}^{h=w_i^1} \hat{r}_h^N \,, \quad i = 1, 2, \ldots, n \,, \tag{57}$$

for some w^1 in W.

For a $\hat{x}^{N-1}$ in D $(\hat{y}^{N-1})$ there are integers $0 = s_0 \leq s_1 \leq \cdots \leq s_n = n$ such that

$$\hat{x}_j^{N-1} = \sum_{i=1+s_{j-1}}^{i=s_j} \hat{y}_i^{N-1}$$

$$= \sum_{i=1+s_{j-1}}^{i=s_j} \sum_{h=1+w_{i-1}^1}^{h=w_i^1} \hat{r}_h^N$$

$$= \sum_{h=1+w_{s_{j-1}}^1}^{h=w_{s_j}^1} \hat{r}_h^N \,, \quad j = 1, 2, \ldots, n \,. \tag{58}$$

Let

$$w_j = w_{s_j}^1$$

so that

$$\hat{x}_j^{N-1} = \sum_{h=1+w_{j-1}}^{h=w_j} \hat{r}_h^N . \tag{59}$$

Total production completed by facility N-1 up to period i is $\sum_{j=1}^{j=i} \hat{x}_j^{N-1}$.

From equation (59)

$$\sum_{j=1}^{j=i} \hat{x}_j^{N-1} = \sum_{j=1}^{j=i} \sum_{h=1+w_{j-1}}^{h=w_j} \hat{r}_h^N$$

$$= \sum_{h=1}^{h=w_i} \hat{r}_h^N \quad . \tag{60}$$

Total demand for facility N-1 up to period i is

$$\sum_{j=1}^{j=i} \hat{y}_j^{N-1} .$$

From equation (57),

$$\sum_{j=1}^{j=i} \hat{y}_j^{N-1} = \sum_{j=1}^{j=i} \sum_{h=1+w_{j-1}^1}^{h=w_j^1} \hat{r}_h^N$$

$$= \sum_{h=1}^{h=w_i^1} \hat{r}_h^N \quad . \tag{61}$$

It is obvious that

$$\sum_{j=1}^{j=i} \hat{x}_j^{N-1} \geq \sum_{j=1}^{j=i} \hat{y}_j^{N-1}$$

or from equations (60) and (61)

$$\sum_{h=1}^{h=w_i} \hat{r}_h^N \geq \sum_{h=1}^{h=w_i^1} \hat{r}_h^N \quad . \tag{61a}$$

To ensure that equation (61a) holds, it is sufficient to have $w_i \geq w_i^1$. But by equation (56), from the same reasoning,

$$w_i^1 \geq i - \alpha_N \, . \tag{62}$$

We have seen

$$0 = s_0 \leq s_1 \leq \dots \leq s_n = n \, .$$

Similarly

$$0 = w_0^1 \leq w_1^1 \leq \dots \leq w_n^1 = n \, . \tag{63}$$

By equations (62) and (63) if w^1 in W, then w is also in W. It immediately follows, for any dominant $\hat{x}^{N-1}$,

$$\hat{x}_i^{N-1} = \sum_{h=1+w_{i-1}}^{h=w_i} \hat{r}_h^N \, , \quad i = 1, 2, \dots, n$$

for some w in W.

Clearly the above argument can be extended by induction to show that for any dominant $\hat{x}^j$,

$$\hat{x}_i^j = \sum_{h=1}^{h=w_j} \hat{r}_h^N \, , \quad i = 1, 2, \dots, n \tag{64}$$

for some w in W.

The simple structure of the dominant production vectors $\hat{x}^j$ for the series case is now clear. Once W is specified $\hat{x}^j$ has the form described in equation (64) for some w in W. Correspondingly since $\hat{y}^{j-1} = \hat{x}^j$ the

the dominant $\hat{y}^{j-1}$ also have a simple form.

Using these results the set $D(\hat{y}^j)$ can be briefly described. For any w^1 in W let

$$\bar{D}_j(w^1) = \{w \mid x^j \text{ is in } D(y^j) \text{ where}$$

$$\hat{x}_i^j = \sum_{h=1+w_{i-1}}^{h=w_i} \hat{r}_h^N \text{ and } \hat{y}_i^j = \sum_{h=1+w_{i-1}^1}^{h=w_i^1} \hat{r}_h^N \},$$

$$i = 1, 2, \ldots, n .$$

Given

$$\hat{y}_i^j = \sum_{h=1+w_{i-1}^1}^{h=w_i^1} \hat{r}_h^N, \quad i = 1, 2, \ldots, n$$

for some w^1 in W, then a vector w is in $\bar{D}_j(w^1)$, if and only if the production vector

$$\hat{x}_i^j = \sum_{h=1+w_{i-1}}^{h=w_i} \hat{r}_h^N$$

is in $D(\hat{y}^j)$. From the previous discussion it is clear that $\bar{D}_j(w^1)$ is contained in W for all w^1 in W.

The cost structure can also be expressed using the vectors in W. Let

$P_j^*(w, w^1)$ = Total production and inventory cost in facility j over all periods.

If

$$\hat{x}_i^j = \sum_{h=1+w_{i-1}}^{h=w_i} \hat{r}_h^N$$

and

$$\hat{y}_i^j = \sum_{h=1+w_{i-1}^1}^{h=w_i^1} \hat{r}_h^N , \quad i = 1, 2, \ldots, n .$$

Then

$$p_j^*(w, w^1) = p_j \left(\sum_{h=1+w_0}^{h=w_1} \hat{r}_h^N, \ldots, \sum_{h=w_{n-1}}^{h=n} \hat{r}_h^N \right)$$

$$+ \sum_{i=1}^{i=n} M_i^j \left\{ \sum_{h=1+w_i^1}^{h=w_i} \hat{r}_h^N \right\}. \tag{65}$$

The dynamic programming recursion relation can now be developed.

Let

$$F_j(w^1) = \text{Minimum cost in facilities 1 through } j,$$

if

$$\hat{y}_i^j = \sum_{h=1+w_{i-1}^1}^{h=w_i^1} \hat{r}_h^N, \quad i = 1, 2, \ldots, n$$

for w^1 in W. Then for $1 \leq j \leq N$,

$$F_j(w^1) = \text{Min } \{p_j^*(w, w^1) + F_{j-1}(w)\} \tag{66}$$

for all w^1 in W if $j < N$. If $j = N$,

$$w^1 = (0, 1, 2, 3, \ldots, n-1, n) .$$

Also

$$F_0(w^1) = 0$$

for all w^1.

Following a dominant schedule a dominant $\hat{y}^j$ will have the form

$$\hat{y}_i^j = \sum_{h=1+w_{i-1}^1}^{h=w_i^1} \hat{r}_h^N, \quad i = 1, 2, \ldots, n$$

for some w^1 in W.

Given this $\hat{y}^j$ a dominant $\hat{x}^j$ will have the form

$$\hat{x}_i^j = \sum_{h=1+w_{i-1}}^{h=w_i} \hat{r}_h^N, \quad i = 1, 2, \ldots, n$$

for some w in $\bar{D}_j(w^1)$.

To illustrate the usefulness of this discussion, let us work out a numerical example.

4.2 EXAMPLE 3. THREE FACILITY CASE

Assume there are 3 facilities, so that N = 3.

Let

$$a^{j,j+1} = 1, \quad j = 1, 2.$$

Hence each facility must produce one unit for each unit withdrawn from stock. It is also assumed that $\alpha_N = 0$, which means that facility N cannot backlog. Sales forecast for the product is

Period	1	2	3
Demand	9	16	9 units

The holding cost is $1/unit/period, for each facility. The production cost is given by

$$p_j(x^j) = x_1^j + 8\ (x_2^j + x_3^j)^{1/2}, \quad j = 1, 2, 3.$$

Plan an optimal schedule to minimize the cost.

4.3 SOLUTION BY DYNAMIC PROGRAMMING

By equation (65), the total production and inventory cost is

$$p_j^*(w, w^1) = \sum_{h=1}^{h=w_1} r_h^N + 8 \left\{ \sum_{h=1+w_1}^{h=3} r_h^N \right\}^{1/2} + 1 \sum_{h=1}^{h=3} \sum_{i=1+w_h^1}^{i=w_h} r_h^N ,$$

$$j = 1, 2, 3.$$

The set W is given by

$$W = \{ (w_1^1, w_2^1, w_3^1), (w_1^1, w_2^1, w_3^1) (w_1^1, w_2^1, w_3^1)(w_1^1, w_2^1, w_3^1)(w_1^1, w_2^1, w_3^1)\}$$

$$= \{ (1, 2, 3), (1, 3, 3), (2, 2, 3), (2, 3, 3), (3,3,3)\} .$$

To each w^1 in W corresponds a $\bar{D}_j(w)$ as follows.

$$\bar{D}_j (w_1^1, w_2^1, w_3^1) = \bar{D}_j (1, 2, 3)$$

$$= \{ (1, 2, 3), (1, 3, 3), (2, 2, 3),(2, 3, 3),(3, 3, 3)\}.$$

Similarly

$$\bar{D}_j (1, 3, 3) = \{ (1, 3, 3), (2, 3, 3), (3, 3, 3)\} ,$$

$$\bar{D}_j (2, 2, 3) = \{ (2, 2, 3), (2, 3, 3), (3, 3, 3)\} ,$$

$$\bar{D}_j (2, 3, 3) = \{ (2, 3, 3), (3, 3, 3)\} ,$$

$$\bar{D}_j (3, 3, 3) = \{ (3, 3, 3)\} ,$$

for all j .

Now let us calculate $p_j^*(w, w^1)$ for all values w^1 in W and w in $\bar{D}_j(w^1)$ and all values of j. We will present here sample calculations and then summarize the values of $p_j^*(w, w^1)$ in the form of a table.

For example consider

$$p_j^* \{(2, 3, 3), (2, 2, 3)\}$$

the sum of production cost and inventory cost of j^{th} facility for all the 3 periods.

Here

$$w_1 = 2, \quad w_2 = 3, \quad w_3 = 3,$$

$$w_1^1 = 2, \quad w_2^1 = 2, \quad w_3^1 = 3\ .$$

$w_1^1 = 2$ means that the assumed demand in the first period is equal to the total market requirement up to and including second period. Therefore, assumed demand in the first period is $9 + 16 = 25$ units. Since $w_2^1 = 2 = w_1^1$, the assumed demand in the second period is 0. $w_3^1 = 3$ implies that assumed demand in the third period is market requirement of the third period which is equal to 9 units.

$w_1 = 2$ means that production in the first period is sufficient to satisfy the market requirement up to and including second period and is equal to 25 units. $w_2 = 3$ means that production in the second period is sufficient to satisfy the market requirement of third period and is equal to 9 units. Since $w_3 = 3 = w_2$, it means that there is no production in the third period. We can now interpret the assumed demand and production as follows

Period	Assumed demand	Production level
1	25 units	25 units
2	0 units	9 units
3	9 units	0 units

Now we are in a position to calculate the production and inventory costs.

$$\text{Production cost} = \sum_{h=1}^{h=w_1} r_h^N + 8 \left\{ \sum_{h=1+w_1}^{h=w_n=n} r_h^N \right\}^{1/2}$$

$$= \sum_{h=1}^{h=2} r_h^3 + 8 \left\{ \sum_{h=1+2}^{h=3} r_h^3 \right\}^{1/2}$$

$$= (r_1^3 + r_2^3) + 8\,(r_3^3)^{1/2}$$

$$= (9 + 16) + 8\,(9)^{1/2}$$

$$= 25 + 24$$

$$= 49 .$$

Alternatively, production cost can be calculated by making use of the preceding table. Going through the column under "Production level" we note

$$x_1^j = 25 \qquad x_2^j = 9 \qquad x_3^j = 0 .$$

Hence

$$\text{Production cost } p_j(x^j) = x_1^j + 8\,(x_2^j + x_3^j)^{1/2}$$

$$= 25 + 8\,(9 + 0)^{1/2}$$

$$= 49 .$$

Inventory cost $M_i^j(I_i^j) = \sum_{h=1}^{h=3} \sum_{i=1+w_h^1}^{i=w_h} r_h^N$

$$= r_3^3$$

$$= 9 \ .$$

Alternatively, by looking at the preceding table, we find that 9 units are carried in the second period which incur a cost of $9.

Hence

$$p_j^* \{ (2, 3, 3), (2, 2, 3) \} = 49 + 9$$

$$= 58 \ .$$

By the same reasoning p_j^* (w, w^1) can be calculated for other sets of (w, w^1). The results are summarized as follows:

w	w^1	$p_j^*(w, w^1)$
(1, 2, 3)	(1, 2, 3)	49
(1, 3, 3)	(1, 2, 3)	58
(2, 2, 3)	(1, 2, 3)	65
(3, 3, 3)	(1, 2, 3)	68
(2, 3, 3)	(1, 2, 3)	74
(1, 3, 3)	(1, 3, 3)	49
(2, 3, 3)	(1, 3, 3)	65
(3, 3, 3)	(1, 3, 3)	59
(2, 2, 3)	(2, 2, 3)	49
(2, 3, 3)	(2, 2, 3)	58
(3, 3, 3)	(2, 2, 3)	52
(2, 3, 3)	(2, 3, 3)	49
(3, 3, 3)	(2, 3, 3)	43
(3, 3, 3)	(3, 3, 3)	34

The recursive calculations are summarized below:

$$F_1\,(1, 2, 3) = \min \begin{Bmatrix} p_1^*\{(1, 2, 3)\,(1, 2, 3)\} \\ p_1^*\{(1, 3, 3)\,(1, 2, 3)\} \\ p_1^*\{(2, 2, 3)\,(1, 2, 3)\} \\ p_1^*\{(2, 3, 3)\,(1, 2, 3)\} \\ p_1^*\{(3, 3, 3)\,(1, 2, 3)\} \end{Bmatrix}$$

$$= \min\,(49, 58, 65, 74, 68)$$

$= 49$ with optimal decision (1, 2, 3).

Similarly

$$F_1\ (1, 3, 3) = 49 \text{ with decision } (1, 3, 3)\ ,$$

$$F_1\ (2, 2, 3) = 49 \text{ with decision } (2, 2, 3)\ ,$$

$$F_1\ (2, 3, 3) = 43 \text{ with decision } (3, 3, 3)\ ,$$

$$F_1\ (3, 3, 3) = 34 \text{ with decision } (3, 3, 3)\ ,$$

$$F_2\ (1, 2, 3) = \min \begin{Bmatrix} p_2^* \ \{(1, 2, 3)\ (1, 2, 3)\} + F_1\ (1, 2, 3) \\ p_2^* \ \{(1, 3, 3)\ (1, 2, 3)\} + F_1\ (1, 3, 3) \\ p_2^* \ \{(2, 2, 3)\ (1, 2, 3)\} + F_1\ (2, 2, 3) \\ p_2^* \ \{(2, 3, 3)\ (1, 2, 3)\} + F_1\ (2, 3, 3) \\ p_2^* \ \{(3, 3, 3)\ (1, 2, 3)\} + F_1\ (3, 3, 3) \end{Bmatrix}$$

$$= 98 \text{ with decision } (1, 2, 3)\ .$$

Similarly

$$F_2\ (1, 3, 3) = 93 \text{ with decision } (3, 3, 3)\ ,$$

$$F_2\ (2, 2, 3) = 86 \text{ with decision } (3, 3, 3)\ ,$$

$$F_2\ (3, 3, 3) = 68 \text{ with decision } (3, 3, 3)\ ,$$

$$F_3\ (1, 2, 3) = 136 \text{ with decision } (3, 3, 3)\ .$$

The optimal schedule is thus

$$x^1 = (34, 0, 0)\ ,$$

$$x^2 = (34, 0, 0)\ ,$$

$$x^3 = (34, 0, 0)\ .$$

That means the entire production takes place in the first period for all facilities. The cost corresponding to the optimal schedule is \$136.00

4.4 SOLUTION BY THE DISCRETE MAXIMUM PRINCIPLE

Let us define each facility as a stage. Hence $n = 1, 2, 3$ and $N = 3$. Let

x_1^n = Number of units remaining to be produced by n^{th} facility, at the end of first period, $n = 1, 2, 3$,

x_2^n = Number of units remaining to be produced by n^{th} facility at the end of second period, $n = 1, 2, 3$.

Obviously, number of units remaining to be produced at the end of the third period is zero.

θ_1^n = Number of units produced by the n^{th} facility in the first period,

θ_2^n = Number of units produced by the n^{th} facility in the second period.

It may be noted that the number of units produced in the third period equals 34 minus number of units produced in the first two periods. Hence we will not give it a special nomenclature.

We can write the performance equations as follows:

$$x_1^n = x_1^{n-1} - \theta_1^n , \quad n = 1, 2, 3,$$

$$x_1^0 = 34 \text{ and } 0 \leq x_1^3 \leq 25, \tag{67}$$

$$x_2^n = x_2^{n-1} - \theta_2^n, \quad n = 1, 2, 3,$$

$$x_2^0 = 25 \text{ and } 0 \leq x_2^3 \leq 9 . \tag{68}$$

Let us define a new state variable x_3 to represent cost.

x_3^n = Total cost up to and including n^{th} facility, for all the three periods.

$$x_3^n = x_3^{n-1} + \theta_1^n + 8\ (34 - \theta_1^n)^{1/2} + I_0^n + (\theta_1^n - Q_1^n) + I_1^n + (\theta_2^n - Q_2^n) \qquad (69)$$

$$= x_3^{n-1} + \theta_1^n + 8\ (34 - \theta_1^n)^{1/2} + 2\ (\theta_1^n - Q_1^n) + (\theta_2^n - Q_2^n) \qquad (69a)$$

where

Q_i^n = Demand for the product of the n^{th} facility in the i^{th} period,

I_i^n = Inventory level of n^{th} facility in the i^{th} period

$= I_{i-1}^n + (\theta_i^n - Q_i^n)$

and

$$I_0^n = 0,\ I_3^n = 0\ .$$

The Hamiltonian Function is

$$H^n = \sum_{i=1}^{3} z_i^n x_i^n$$

$$= z_1^n(x_1^{n-1} - \theta_1^n) + z_2^n(x_2^{n-1} - \theta_2^n) +$$

$$z_3^n\ \{\ x_3^{n-1} + \theta_1^n + 8\ (34 - \theta_1^n)^{1/2} + 2\ (\theta_1^n - Q_1^n) + (\theta_2^n - Q_2^n)\ \}. \qquad (70)$$

The objective function to be minimized is

$$S = \sum_{i=1}^{3} c_i\ x_i^3 = x_3^3\ .$$

Hence

$$c_1 = 0 \ , \ c_2 = 0, \ c_3 = 1.$$

Since x_1^3 and x_2^3 are fixed

$$z_1^3 \neq c_1, \quad z_2^3 \neq c_2, \text{ but } z_3^3 = c_3 = 1 \ .$$

The adjoint variables are

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = z_1^n \ ,$$

$$z_2^{n-1} = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n \ ,$$

$$z_3^{n-1} = \frac{\partial H^n}{\partial x_3^{n-1}} = z_3^n \ .$$

Since

$$z_3^3 = 1,$$

We have

$$z_3^n = 1, \quad n = 1, 2, 3 \ .$$

Substituting this in H^n of equation (70), we obtain

$$H^n = z_1^n \ (x_1^{n-1} - \theta_1^n) + z_2^n \ (x_2^{n-1} - \theta_2^n) + \{ \ x_3^{n-1} + \theta_1^n$$

$$+ 8 \ (34 - \theta_1^n)^{1/2} + 2 \ (\theta_1^n - Q_1^n) + (\theta_2^n - Q_2^n) \} \ . \tag{71}$$

Since z_1^n, x_1^{n-1}, x_2^{n-1}, x_3^{n-1}, Q_1^n, and Q_2^n in equation (71) are constants,

we write the variable portion of H^n as

$$H_v^n = -z_1^n\theta_1^n - z_2^n\theta_2^n + \theta_1^n + 8\ (34 - \theta_1^n)^{1/2} + 2\ \theta_1^n + \theta_2^n$$

$$= \theta_1^n\ (3 - z_1^n) + 8\ (34 - \theta_1^n)^{1/2} + \theta_2^n\ (1 - z_2^n)\ . \qquad (72)$$

Again H_v^n can be represented as

$$H_v^n = (H_v^n)_{\theta_1} + (H_v^n)_{\theta_2}$$

where

$$(H_v^n)_{\theta_1} = \theta_1^n\ (3 - z_1^n) + 8\ (34 - \theta_1^n)^{1/2} \qquad (73)$$

and

$$(H_v^n)_{\theta_2} = \theta_2^n\ (1 - z_2^n)\ . \qquad (74)$$

S is a minimum when H_v^n is a minimum that is when

$$(H_v^n)_{\theta_1} + (H_v^n)_{\theta_2}$$

is a minimum. We will evaluate the condition for H_v^n to be a minimum by trial and error, since we do not know whether z_1^n and z_2^n are positive or negative. But we do know that z_1^n, and z_2^n are not equal to zero.

Looking at equation (74) we find that $(H_v^n)_{\theta_2}$ is a minimum on the boundary of θ_2^n. θ_2^n can take 3 values namely 25, 16, 0. (We assume that production takes place in batches). So, when $(H_v^n)_{\theta_2}$ is a minimum, θ_2^n can be

two values, 25 or 0, depending on whether $(1 - z_2^n)$ is negative or positive. Corresponding to the values of θ_2^n and depending on the value of $(3 - z_1^n)$, we have to choose the values of θ_1^n by trial and error which minimize $(H_v^n)_{\theta_1}$. Consequently our search has been narrowed down to 3 cases

$$\text{case 1:} \quad \theta_2^n = 0, \quad \theta_1^n = 34 ,$$

$$\text{case 2:} \quad \theta_2^n = 25, \quad \theta_1^n = 9 ,$$

$$\text{case 3:} \quad \theta_2^n = 0, \quad \theta_1^n = 25, \quad n = 1,2,3 .$$

The easiest way of optimum seeking is to substitute the 3 cases in the cost equation and adopt that case which gives minimum cost. Care should be taken while substituting in the cost equation. Since third facility supplies the market, Q_i^n is actual demand for n = 3, and Q_i^n is production level of $(n + 1)^{th}$ facility that equals θ_i^{n+1}, n = 1, 2 .

Case 1: $\theta_1^n = 34$, $\theta_2^n = 0$, for n = 1, 2, 3.

Hence $Q_1^1 = \theta_1^2 = 34$,

$$Q_1^2 = \theta_1^3 = 34,$$

$$Q_1^3 = \text{Actual demand} = 9 .$$

Substituting in equation (69a) yields

$$x_3^1 = 0 + 34 + 8\,(34 - 34)^{1/2} + 2\,(34 - 34) + (0 - 0)$$

$$= 34,$$

$$x_3^2 = 34 + 34 + 8\ (34 - 8)\ (34 - 34)^{1/2} + 2\ (34 - 34) + (0 - 0)$$

$$= 68,$$

$$x_3^3 = 68 + 34 + 8\ (34 - 34)^{1/2} + 2\ (34 - 9) + (0 - 16) \tag{75}$$

$$= 136\ .$$

Case 2: $\theta_1^n = 9, \quad \theta_2^n = 25, \quad n = 1, 2, 3.$

Here $Q_1^1 = \theta_1^2 = 9, \quad Q_2^1 = \theta_2^2 = 25,$

$$Q_1^2 = \theta_1^3 = 9, \quad Q_2^2 = \theta_2^3 = 25,$$

$$Q_1^3 = 9, \qquad Q_2^3 = 16\ .$$

substituting in equation (69a), we obtain

$$x_3^1 = 0 + 9 + 8\ (34 - 9)^{1/2} + 2\ (9 - 9) + (25 - 25)$$

$$= 49,$$

$$x_3^2 = 49 + 9 + 8\ (34 - 9)^{1/2} + 2\ (9 - 9) + (25 - 25)$$

$$= 98,$$

$$x_3^2 = 98 + 9 + 8\ (34 - 9)^{1/2} + 2\ (9 - 9) + (25 - 16)$$

$$= 98 + 9 + 40 + 9$$

$$= 156. \tag{76}$$

Case 3: Similarly we can calculate the cost in this case = 163. (77)

Comparing equations (75), (76) and (77), case 1 is cheapest.

Hence

$$\theta_1^n = 34, \quad \theta_2^n = 0,$$

that is, the optimal production schedule for the 3 facilities is

	Production level in period		
Facility	1	2	3
1	34	0	0 units
2	34	0	0 units
3	34	0	0 units

total minimum cost = $136.00 .

COMMENT: Comparing the two methods namely dynamic programming and the discrete maximum principle to solve the series facility case, we observe that the area of search is quite wide for dynamic programming, whereas, for the discrete maximum principle it is narrowed down to only three cases.

4.5 EXAMPLE 4. FOUR FACILITY CASE

Assume that there are four facilities connected in series, so that $N = 4$. Let $a^{j,j+1} = 1$, $j = 1, 2, 3,$ hence each facility must produce one unit for each unit withdrawn from stock. It is assumed that $\alpha_4 = 0$, which means facility 4 cannot backlog. Facility 4 must satisfy the following demand

period	1	2	3
demand	9	16	9 units

over three periods.

The cost of producing and holding stock is the same in each facility and is as follows:

Production cost

$$p_j^*(x^j) = x_1^j + 8\,(x_2^j + x_3^j)^{1/2} \quad , \quad j = 1, 2, 3, 4 \ .$$

Holding cost

$$M_i^j\,(I_i^j) = 2I_i^j, \quad j = 1, 2, 3, 4; \quad i = 1, 2, 3 \ .$$

Plan an optimum production schedule.

4.6 SOLUTION BY DYNAMIC PROGRAMMING

The set W = {(1, 2, 3), (1, 3, 3), (2, 2, 3) (2, 3, 3), (3, 3, 3)} .

To each w^1 in W corresponds a $\bar{D}_j(w)$ as follows:

$$\bar{D}_j\ (1, 2, 3) = \{\ (1, 2, 3),\ (1, 3, 3),\ (2, 2, 3)$$
$$(2, 3, 3),\ (3, 3, 3)\ \},$$

$$\bar{D}_j\ (1, 3, 3) = \{\ (1, 3, 3),\ (3, 3, 3)\ \},$$

$$\bar{D}_j\ (2, 2, 3) = \{\ (2, 2, 3),\ (2, 3, 3),\ (3, 3, 3)\}\ ,$$

$$\bar{D}_j\ (2, 3, 3) = \{\ (2, 3, 3),\ (3, 3, 3)\}\ ,$$

$$\bar{D}_j\ (3, 3, 3) = \{\ (3, 3, 3)\ \}\ ,$$

for all j.

Now let us calculate $p_j^*\ (w, w^1)$ for all values of w^1 in W and w in $\bar{D}_j\ (w^1)$ and for all j. We will present here sample calculations and

summarize the values of $p_j^* (w, w^1)$ in a table. For example, consider

$$p_j^* (w, w^1) = p_j^* \{ (w_1, w_2, w_3) (w_1^1, w_2^1, w_3^1) \}$$

$$= p_j^* \{ (2, 3, 3), (2, 2, 3) \} .$$

$w_1^1 = 2$ means that the assumed demand in the first period is equal to the total market requirement up to and including the second period. $w_2^1 = 2$ means that assumed demand in the second period is equal to zero. Since $w_1^1 = 2$, $w_3^1 = 3$ means that assumed demand in the third period is equal to the actual demand of the third period. $w_1 = 2$ means that the production that takes place in the first period is sufficient to satisfy demand up to and including the second period. $w_2 = 3$ means that production in the second period is sufficient to satisfy the demand of the third period, so that there is no production in the third period. Hence

$$p_j^* \{ (2, 3, 3), (2, 2, 3) \} = 1 \ (9 + 16) + 8 \ (9)^{1/2} + 2 \times 9 = 67 .$$

Similarly we can calculate $p_j^* (w, w^1)$ for other values of w and w^1 and tabulate the results as follows:

w	w^1	$p_j^*(w, w^1)$
(1, 2, 3)	(1, 2, 3)	49
(1, 3, 3)	(1, 2, 3)	67
(2, 2, 3)	(1, 2, 3)	81
(2, 3, 3)	(1, 2, 3)	99
(3, 3, 3)	(1, 2, 3)	102
(1, 3, 3)	(1, 3, 3)	49
(3, 3, 3)	(1, 3, 3)	84
(2, 2, 3)	(2, 2, 3)	49
(2, 3, 3)	(2, 2, 3)	67
(3, 3, 3)	(2, 2, 3)	80
(2, 3, 3)	(2, 3, 3)	49
(3, 3, 3)	(2, 3, 3)	67
(3, 3, 3)	(3, 3, 3)	34

The recursion calculations are summarized below.

$$F_1\,(1,2,3) = \min\{\, p_1^*\,\{(1,2,3)\,(1,2,3)\},\; p_1^*\,\{(1,3,3)\,(1,2,3)\},\; p_1^*\,\{(2,2,3)\,(1,2,3)\},\; p_1^*\,\{(2,3,3)\,(1,2,3)\},\; p_1^*\,\{(3,3,3)\,(1,2,3)\}\}$$

$$= \min\{\,49,\ 67,\ 81,\ 99,\ 102\,\}$$

$$= 49 \text{ with decision } (1,2,3)\ .$$

Similarly,

$$F_1\,(1,3,3) = 49 \text{ with decision } (1,3,3),$$

$$F_1\,(2,2,3) = 49 \text{ with decision } (2,2,3),$$

$$F_1\,(2,3,3) = 49 \text{ with decision } (2,3,3),$$

$$F_1\,(3,3,3) = 34 \text{ with decision } (3,3,3)\ .$$

$$F_2\,(1,2,3) = \min\{\, p_2^*\,\{(1,2,3),(1,2,3)\} + F_1\,(1,2,3),\; p_2^*\,\{(1,3,3),(1,2,3)\} + F_1\,(1,3,3),\; p_2^*\,\{(2,2,3),(1,2,3)\} + F_1\,(2,2,3),\; p_2^*\,\{(3,3,3),(1,2,3)\} + F_1\,(3,3,3)\,\}$$

$$= 98 \text{ with decision } (1,2,3)\ .$$

Similarly,

$F_2\ (1, 3, 3) = 98$ with decision $(1, 3, 3)$,

$F_2\ (2, 2, 3) = 98$ with decision $(2, 2, 3)$,

$F_2\ (2, 3, 3) = 98$ with decision $(2, 3, 3)$,

$F_2\ (3, 3, 3) = 68$ with decision $(3, 3, 3)$,

$$\begin{aligned} F_3\ (1, 2, 3) = \min \{ & p_3^* \ \{(1, 2, 3), (1, 2, 3)\} + F_2\ (1, 2, 3), \\ & p_3^* \ \{(1, 3, 3), (1, 2, 3)\} + F_2\ (1, 3, 3), \\ & p_3^* \ \{(2, 2, 3), (1, 2, 3)\} + F_2\ (2, 2, 3), \\ & p_3^* \ \{(2, 3, 3), (1, 2, 3)\} + F_2\ (2, 3, 3), \\ & p_3^* \ \{(3, 3, 3), (1, 2, 3)\} + F_2\ (3, 3, 3)\} \end{aligned}$$

$= 147$ with decision $(1, 2, 3)$.

Similarly,

$F_3\ (1, 3, 3) = 147$ with decision $(1, 3, 3)$,

$F_3\ (2, 2, 3) = 147$ with decision $(2, 2, 3)$,

$F_3\ (2, 3, 3) = 147$ with decision $(2, 3, 3)$,

$F_3\ (3, 3, 3) = 102$ with decision $(3, 3, 3)$.

For the last facility, that is, facility 4, the actual demand is $w = (1, 2, 3)$, therefore, we obtain the total minimum cost from $F_4\ (1, 2, 3)$ as follows

$$
\begin{aligned}
F_4(1, 2, 3) = \min \{ & p_4^* \{(1, 2, 3)(1, 2, 3)\} + F_3(1, 2, 3) , \\
& p_4^* \{(1, 3, 3)(1, 2, 3)\} + F_3(1, 3, 3) , \\
& p_4^* \{(2, 2, 3)(1, 2, 3)\} + F_3(2, 2, 3) , \\
& p_4^* \{(2, 3, 3)(1, 2, 3)\} + F_3(2, 3, 3) , \\
& p_4^* \{(3, 3, 3)(1, 2, 3)\} + F_3(3, 3, 3)\}
\end{aligned}
$$

$$= \min \{196, 214, 214, 214, 204\}$$

$$= 196 \text{ with decision } (1, 2, 3) .$$

The optimal schedule is thus

$$x^1 = (9, 16, 9) ,$$

$$x^2 = (9, 16, 9),$$

$$x^3 = (9, 16, 9),$$

$$x^4 = (9, 16, 9) .$$

Total cost = \$196 .

4.7 EXAMPLE 5. FIVE FACILITY CASE AND SOLUTION BY DYNAMIC PROGRAMMING

Work out the above problem assuming that there are five facilities and compare the results.

For the 5 facility situation, the computational procedure up to and including facility 3 is the same as for 4 facility situation. For the fourth facility we have to calculate $F_4(1, 2, 3)$, $F_4(1, 3, 3)$, $F_4(2, 2, 3)$, $F_4(2, 3, 3)$ and $F_4(3, 3, 3)$. We have seen in the previous example that

$F_4(1, 2, 3) = 196$ with decision (1, 2, 3).

Similarly

F_4 (1, 3, 3) = 186 with decision (3, 3, 3) ,

F_4 (2, 2, 3) = 182 with decision (3, 3, 3),

F_4 (2, 3, 3) = 169 with decision (3, 3, 3),

F_4 (3, 3, 3) = 136 with decision (3, 3, 3) .

For the last facility, that is facility 5, the actual demand is given by

$$w = (1, 2, 3) .$$

Therefore, we obtain the total minimum cost from F_5 (1, 2, 3).

$$\begin{aligned} F_5 (1, 2, 3) = \text{min of } \{ & p_5^* \{(1, 2, 3), (1, 2, 3)\} + F_4 (1, 2, 3), \\ & p_5^* \{(1, 3, 3), (1, 2, 3)\} + F_4 (1, 3, 3), \\ & p_5^* \{(2, 2, 3), (1, 2, 3)\} + F_4 (2, 2, 3), \\ & p_5^* \{(2, 3, 3), (1, 2, 3)\} + F_4 (3, 3, 3), \\ & p_5^* \{(3, 3, 3), (1, 2, 3)\} + F_4 (3, 3, 3)\} \end{aligned}$$

$$= \min \{ 245, 253, 263, 268, 238\}$$

$$= 238 \text{ with decision } (3, 3, 3) .$$

Hence the optimal schedule is

$x^1 = (34, 0, 0)$,

$x^2 = (34, 0, 0)$,

$x^3 = (34, 0, 0)$,

$x^4 = (34, 0, 0)$,

$x^5 = (34, 0, 0)$.

Total cost = \$238 .

Comments - Comparing the results of the above two examples, we find that if the number of facilities is equal to or less than four, then it is not profitable to carry inventory. If the number of facilities exceeds 4, it is profitable to produce in the first period only and carry inventory. This is because, for the 5-facility case, the inventory cost is counterbalanced by production costs. We find that the reduction in production costs is more than increase in inventory cost, when N increases from 4 to 5.

5. A DETERMINISTIC MULTIPERIOD PRODUCTION SCHEDULING MODEL WITH BACKLOGGING

As before, a deterministic multi-period production and inventory model that has concave production costs and piecewise concave inventory costs is analyzed. An essential feature of this model is that it permits backlogging of unsatisfied demand, otherwise the model is similar to one discussed before. Permitting backlogging, mathematically means to permit negative inventories for which penalty has to be paid. The cost function is piecewise concave.

5.1 PROBLEM STATEMENT (9)

The problem can be stated as follows: Given certain fixed non-negative market requirements r, find a production schedule k that minimizes the piecewise concave function

$$F(k) = p(k) + \sum_{h=1}^{h=n} M_h(k) \tag{78}$$

subject to

$$I_i = \sum_{h=1}^{h=i} (x_h - r_h), \qquad i = 1, 2, \ldots, n \,, \tag{79}$$

$$I_i \geq - \sum_{h=i-\alpha+1}^{h=i} r_h \,, \qquad i = 1, 2, \ldots, n \,, \tag{80}$$

$$I_n = 0, \tag{81}$$

$$x_i \geq 0 \,, \qquad i = 1, 2, \ldots, n \,, \tag{82}$$

$$x_i = 0 \,, \text{ for } i \leq 0$$

and

$$r_i = 0, \text{ for } i \leq 0$$

Let x be the set of all vectors k that satisfy the constraint system given by equations (79), (80), (81) and (82). Any k in x is called a feasible production vector. A k in x that minimizes F(k) on x is called optimal production vector. It should be noted that x is closed, bounded polyhedral and thus is a compact convex set.

A production schedule k is said to satisfy exact requirements if there are integers $0 = s_0 \leq s_1 \leq \cdots \_ s_n = n$ such that

$$x_i = \sum_{h=1+s_{i-1}}^{h=s_i} r_h \,, \quad i = 1, 2, \ldots, n \,. \tag{83}$$

All vectors in the dominant set and hence an optimal production schedule must satisfy exact requirements. In an exact requirement schedule, the production completed in period 1 satisfies market requirements exactly through s_1. If $s_1 = 0$ then $x_1 = 0$ and zero periods are exactly satisfied. By the beginning of period i production has been completed to exactly satisfy requirements through period s_{i-1}. The production completed in period i will satisfy market requirements through period s_i.

The backlogging assumption is satisfied if $s_i \geq i - \alpha$. This requirement insures that by the end of period i production must have been completed to satisfy requirements through period $i-\alpha$. Equivalently the requirements on s_i insure that equation (80) is satisfied.

5.2 AN ALGORITHM BASED ON DYNAMIC PROGRAMMING (9)

Having defined the dominant set and the integer, we will go on to derive an algorithm to solve this problem. There are a large number of vectors in the dominant set. The type of algorithm that should be employed is heavily dependent upon the cost structure. In this section an algorithm

for a commonly found cost structure will be developed. It is assumed that the concave cost term p(k) can be expressed as

$$p(k) = \sum_{i=1}^{i=n} p_i(x_i)$$

Where $p_i(x_i)$ is the cost of completing production x_i in period i. The total cost of producing schedule k is then

$$F(k) = \sum_{i=1}^{i=n} \{p_i(x_i) + M_i(I_i)\}$$

where $M_i(I_i)$ is the cost of inventory. More meaning will be given to this, as the discussion proceeds. A dynamic programming algorithm will be used to find the optimal schedule for the above cost function. The algorithm makes use of the structure of the dominant schedules. Since the dominant schedules satisfy exact requirements the inventory in period i can be expressed in terms of an integer s as follows:

$$I_i = \sum_{h=i+1}^{h=s} r_h \quad . \tag{84}$$

The integer s specifies that a stock on hand is sufficient to satisfy requirements through period s. When the inventory equation can be expressed as in equation (84) the inventory horizon is said to be s. The inventory charges, $M_i(I_i)$, can also be expressed as a function of s. Let

$$M_i \sum_{h=i+1}^{h=s} r_h = M_i^*(s) \ .$$

In a schedule following exact requirements, the production amounts satisfy exact requirements also.

If

$$I_{i-1} = \sum_{h=i}^{h=s} r_h ,$$

then

$$x_i = \sum_{h=s+1}^{h=t} r_h$$

for some integer $t \geq s$ and the production completed in period i satisfies requirements from periods $s + 1$ to t. The integers s and t specify the production completed in period i.

Let

$$p_i \sum_{h=s+1}^{h=t} r_h = p_i^* (s, t)$$

which specifies the production charges in terms of s and t. It may be noted that if the inventory level in period $i - 1$, is s and the production completed in period i satisfies requirements from periods $s + 1$ to t then the inventory horizon at the end of period i is t.

With the above notation, it is possible to develop the dynamic programming recursive equation. Let $F_i(s)$ be the minimum cost from period i through n when following an optimal production schedule in period i through n, given that the inventory horizon at the end of period $i - 1$ is s.

The recursive relationship for $1 < i < n$ is

$$F_i(s) = p_i^* (s, s) + M_i^* (s) + F_{i+1} (s) \text{ if } s \geq i$$

$$= \text{Min } \{ p_i^* (s, t) + M_i^* (t) + F_{i+1} (t) \mid n \geq t \geq \max (s, i-\alpha)\} \text{ if } s < i$$

for all s such that $\max(0, i-1-\alpha) \leq s \leq n$.

For periods i, $1 < i < n$, if the inventory in period i-1 is s then the relation $n \geq s \geq \max(0, i-1-\alpha)$ should be satisfied to insure feasibility and prevent excess backlogging. If $s \geq i$, then there is positive inventory and following a dominant schedule x_i must be zero. x_i then actually supplies requirements from period i + 1 to s. The production charge in period i is $p_i^*(s, s)$ and the inventory charge is $M_i^*(s)$. Since the inventory horizon in period i is s, the minimum cost from period i + 1 to the end is $F_{i+1}(s)$.

If $s < i$ then $x_i > 0$ is permissible following a dominant schedule. If the production completed in period i supplies requirements from periods s + 1 to t, then in order to maintain feasibility and prevent excess backlogging, we must have $n \geq t \geq \max(s, i-\alpha)$.

The charges in period i are $p_i^*(s, t) + M_i^*(t)$ because the production completed in period i satisfies requirements from periods s + 1 to t and in the period i the inventory horizon is t.

For period i = 1 and i = n the recursive equations require the following modifications to insure the initial and final inventories are zero.

$$F_1(0) = \min \{ p_i^*(0, t) + M_i^*(t) + F_2(t) \} ,$$

$$n \geq t \geq \max(i-\alpha, 0)$$

and

$$F_n(s) = p_n^*(s, n) + M_n^*(n) .$$

5.3 EXAMPLE 6. SINGLE FACILITY CASE WITH BACKLOGGING

To illustrate the use of the algorithm consider the following example. Assume that it is necessary to supply market requirements over the next three

periods of $r_1 = 3$, $r_2 = 4$ and $r_3 = 2$. A one period backlog is permitted and all requirements must be satisfied by the end of period 3, so that $\alpha = 1$ and $I_3 = 0$.

The cost $p_i(x_i)$ of producing x_i units in period i is

$$p_i(x_i) = \delta_i\,(x_i) + 3x_i, \quad i = 1, 2, 3,$$

where

$$\delta_i(0) = 0 \quad \text{for all } i,$$

$$\delta_1(x_1) = 7 \quad \text{if } x_1 > 0,$$

$$\delta_2(x_2) = 4 \quad \text{if } x_2 > 0,$$

$$\delta_3(x_3) = 3 \quad \text{if } x_3 > 0\ .$$

The production cost in each period is characterized by a set up charge $\delta_i(x_i)$ plus a linear production cost. The inventory cost $M_i(I_i)$ is \$1 per unit per period and shortage cost is \$2 per unit per period.

Find an optimum schedule that minimizes cost.

5.4 SOLUTION BY DYNAMIC PROGRAMMING

We notice that the holding cost for positive inventory is linear at one dollar per unit per period while the shortage cost is also linear but at \$2.00 per unit backlogged per period.

The problem is to find the production schedule, $k = (x_1\ x_2\ x_3)$, that minimizes cost. The dominant set for these requirements consist of

$2^3 = 8$ schedules as follows:

$$D = \{(3, 4, 2), (7, 0, 2), (3, 6, 0), (9, 0, 0), (0, 3, 6),$$

$$(0, 9, 0), (0, 7, 2), (3, 0, 6)\}.$$

It may be noted that while calculating the cost corresponding to each schedule, for convenience, we could ignore the linear portion of the production costs, since it is identical in each period. Only the set up charges and the holding cost will be considered.

We have proved

$$F_n(s) = p_n^* (s, n) + M_n^* (n),$$

$$F_3(1) = p_3^* (1, 3) + M_3^* (3)$$

$$= 3 + 0 = 3 .$$

Similarly

$$F_3^* (2) = 3,$$

$$F_3^* (3) = 0.$$

$F_2(0)$ = Minimum of the following:

$$p_2^* (0, 1) + M_2^* (1) + F_3 (1) = 4 + 8 + 3 = 15,$$

$$p_2^* (0, 2) + M_2^* (2) + F_3 (2) = 4 + 0 + 3 = 7,$$

$$p_2^* (0, 3) + M_2^* (3) + F_3 (3) = 4 + 2 + 0 = 6,$$

Hence

$F_2(0) = 6$ with decision (3) .

$F_2(1)$ = min of the following:

$$p_2^* (1, 1) + M_2^* (1) + F_3 (1) = 8 + 3 = 11,$$

$$p_2^* (1, 2) + M_2^* (2) + F_3 (2) = 4 + 0 + 3 = 7,$$

$$p_2^* (1, 3) + M_2^* (3) + F_3 (3) = 4 + 2 + 0 = 6.$$

Hence $F_2(1) = 6$ with decision (3).

F_2 (2) = Minimum of the following:

$$p_2^* (2, 2) + M_2^* (2) + F_3 (2) = 3,$$

$$p_2^* (2, 3) + M_2 (3) + F_3 (3) = 6 .$$

Hence

$F_2(2) = 3$ with decision (2) .

$$F_2(3) = p_2^* (3, 3) + H_2^* (3) + F_3(3)$$

$$= 0 + 2 + 0 = 2 \text{ decision } (3) .$$

$F_1(0)$ = Minimum of the following:

$$p_1^* (0, 0) + M_2^* (0) + F_2(0) = 0 + 6 + 6 = 12,$$

$$p_1^* (0, 1) + M_2^* (1) + F_2(1) = 7 + 0 + 6 + 13,$$

$$p_1^* (0, 2) + M_2^* (2) + F_2(2) = 7 + 4 + 3 = 14,$$

$$p_1^* (0, 3) + M_2^* (3) + F_2(3) = 7 + 6 + 0 = 13 .$$

Hence

$$F_1(0) = 12 \text{ with decision } (0) .$$

This means that the entire production takes place in the second period. Hence the optimal production schedule is (0, 9, 0). Minimum cost corresponding to this schedule = 12 + 3 x 9 = \$39.00.

5.5 SOLUTION BY THE DISCRETE MAXIMUM PRINCIPLE

Let us define each period as a stage.

Let

x_1^n = Total number of units produced up to and including n^{th} period, n = 1, 2, 3.

θ^n = Number of units produced in the n^{th} period only.

F^n = Fixed cost in the n^{th} period.

x_2^n = Total cost up to and including n^{th} period, n = 1, 2, 3.

$G(x_1^{n-1}, \theta^n)$ = Cost for n^{th} period only.

Q^n = Demand in the n^{th} period.

We can write the performance equations as follows

$$x_1^n = x_1^{n-1} + \theta^n , \quad n = 1, 2, 3, \tag{72}$$

$$x_1^0 = 0 \text{ and } x_1^3 = 9,$$

$$x_2^n = x_2^{n-1} + G(x_1^{n-1}, \theta^n)$$

$$= x_2^{n-1} + F^n + 3\theta^n + P\,(x_1^n - \sum_{i=1}^{i=n} Q^i), \; n = 1, 2, 3, \tag{73}$$

$$x_2^0 = 0 \ .$$

Where

$P(x_1^n - \sum_{i=1}^{i=n} Q^i)$ represents either inventory cost or shortage cost (that is penalty for backlogging) depending on whether stock on hand is more than or less than the demand.

If $(x_1^n - \sum_{i=1}^{i=n} Q^i) > 0$ then $P = 1$ = inventory cost. If $(x_1^n - \sum_{i=1}^{i=n} Q^i) < 0$ then $P = -2$ = shortage cost. In both cases, the product $P(x_1^n - \sum_{i=1}^{i=n} Q^i)$ is positive. The objective function to be minimized is

$$S = c_1 x_1^3 + c_2 x_2^3 = x_2^3 \ .$$

Hence

$$c_1 = 0 \text{ and } c_2 = 1.$$

The Hamiltonian and adjoint variables are

$$H^n = z_1^n x_1^n + z_2^n x_2^n$$

$$= z_1^n (x_1^{n-1} + \theta^n) + z_2^n \{ x_2^{n-1} + F^n + 3\theta^n + P(x_1^n - \sum_{i=1}^{i=n} Q^i) \} \ ,$$

$$n = 1, 2, 3, \qquad (74)$$

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = z_1^n + P z_2^n \ ,$$

and

$$z_2^{n-1} = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n \quad .$$

Since

$$z_2^3 = c_2 = 1,$$

we have

$$z_2^n = 1, \quad n = 1, 2, 3 .$$

Then

$$z_1^{n-1} = z_1^n + P,$$

and the Hamiltonian becomes,

$$H^n = z_1^n(x_1^{n-1} + \theta^n) + x_2^{n-1} + F^n + 3\theta^n + P(x_1^n - \sum_{i=1}^{i=n} Q^i) \quad . \tag{75}$$

Objective function S is a minimum when H^n is a minimum. We note that z_1^n, x_1^{n-1}, and x_2^{n-1} in H^n are constants and

$$P(x_1^n - \sum_{i=1}^{i=n} Q^i)$$

is always positive. Therefore the variable portion of the Hamiltonian, H_v^n, to be minimized can be written as

$$\begin{aligned} H_v^n &= z_1^n\theta^n + F^n + 3\theta^n \\ &= \theta^n(z_1^n + 3) + F^n \quad . \end{aligned} \tag{76}$$

It may be noted that F^n is zero if θ^n is 0, and is positive constant if

θ^n is positive.

Since H_v^n is a linear in θ^n, it is a minimum on the boundary of admissible values of θ^n. We do not know whether $(z_1^n + 3)$ is negative or positive. If $(z_1^n + 3)$ is negative θ^n should take the maximum value and if $(z_1^n + 3)$ is positive, θ^n should take the minimum value namely zero, in order to minimize H_v^n .

Stage 1:

Substituting n = 1 in equation (76) yields

$$H_v^1 = \theta^1 (z_1^1 + 3) + F^1 .$$

θ^1 can take one of the two values to minimize H_v^1

$$\theta^1 = 0 \text{ or } 9 .$$

Stage 2:

Substituting n = 2 in equation (76) yields

$$H_v^2 = \theta^2 (z_1^2 + 3) + F^2 ,$$

$$\theta^2 = 0 \text{ or } 9.$$

Stage 3:

Substituting n = 3 in equation (76) yields

$$H_v^3 = \theta^3 (z_1^3 + 3) + F^3 ,$$

$$\theta^3 = 0 \text{ or } 6 .$$

By looking at H_v^1, H_v^2 and H_v^3, we find that our search is narrowed down to 2 cases

Case 1:

$$\theta^1 = 0, \quad \theta^2 = 9, \quad \theta^3 = 0 .$$

Case 2:

$$\theta^1 = 9, \quad \theta^2 = 0, \quad \theta^3 = 0 .$$

The easiest way of optimum seeking is to calculate the total costs corresponding to these two cases and adopt that case which gives the minimum cost.

Case 1:

$$\theta^1 = 0, \quad \theta^2 = 9, \quad \theta^3 = 0 .$$

Substituting n = 1 in equation (73) yields

$$x_2^1 = x_2^0 + F^1 + 3\theta^1 + P\,(x_1^1 - \sum_{i=1}^{i=1} Q^i)$$

$$= 0 + 0 + 3x0 + (-2)\,(0 - 3)$$

$$= 6,$$

$$x_2^2 = x_2^1 + F^2 + 3\theta^2 + P(x^2 - \sum_{i=1}^{i=2} Q^i)$$

$$= 6 + 4 + 3x9 + 1\,\{\,9 - (3 + 4)\}$$

$$= 39 .$$

Similarly

$$x_2^3 = 39 + 0 + 3x0 + 0$$

$$= 39 .$$

Total cost = x_2^3 = \$39.

Case 2:

$$\theta_1^1 = 9, \quad \theta^2 = 0, \quad \theta^3 = 0 .$$

$$x_2^1 = x_2^0 + F^1 + 3\theta^1 + p\ (x_1^1 - \sum_{i=1}^{i=1} Q^i)$$

$$= 0 + 7 + 3x9 + 1\ (9-3)$$

$$= 40,$$

$$x_2^2 = 40 + 0 + 0 + 1\ (2)$$

$$= 42,$$

$$x_2^3 = 42 + 0 + 0 + 0$$

$$= 42.$$

Total cost = x_2^3 = \$42 .

Comparing case 1 and case 2 we find that case 1 is cheaper. Hence the optimal production schedule is as follows:

Period	1	2	3
Production level	0	9	0 units

Total cost = \$39.

The same result is obtained by the use of dynamic programming method.

6. APPLICATION OF THE DISCRETE MAXIMUM PRINCIPLE TO LABOUR ASSIGNMENT AS A DYNAMIC CONTROL PROBLEM

In this section, we discuss the near optimal labour assignment with a restriction on the availability of labour and number of machines. The process of labour assignment is formulated as a dynamic control problem. The criterion function employed here is to minimize the total in-process inventory cost over a given time-span. This problem was first solved to obtain necessary and sufficient conditions for the optimal control by the continuous maximum principle by Nelson (7). In this section, an attempt has been made to reassign the labour at discrete time intervals by employing a discrete version of the maximum principle.

The problem in the nutshell, can be stated as follows:

There are L laborers and m machine centers. Each machine center $i = 1, 2, \ldots, m$ consists of f_i identical machines. We assume,

$$L < \sum_{i=1}^{i=m} f_i$$

so that labour is a limiting resource.

Let

λ = Rate of arrival of work to machine center in work units per period.

μ_i = Service rate in work units per period for each machine in machine center i when there is a labourer assigned to the machine i, $i = 1, 2, \ldots, m$.

x_i^n = Queue length at machine center i at the n^{th} period measured in work units, $i = 1, 2, \ldots, m$.

k_i = Inventory cost per work unit per period at machine center i, $i = 1, 2, \ldots, m$.

A job lot is a block of successively arriving work characterized by identical processing requirements. Each job lot requires processing at a completely ordered sequence of machine centers. Both the job routings and service time requirements are known in advance.

The work force is completely homogeneous and flexible, i.e., every labourer is equally efficient at any given machine center. Only one labourer can work on a machine at one time.

Work is processed at each machine center at discrete time intervals. The service rate of the machine center in any period is proportional to the number of labourers assigned to the machine center in that period. The queue discipline is arbitrary except that only one job lot can be processed in any machine center at one time. The portion of a job lot that has been processed instantaneously enters the appropriate queue for its succeeding operation.

As stated before, the problem is to find a labour assignment procedure that minimizes total in-process inventory costs over the n time periods.

Let us denote the system state vector of queue lengths in the n^{th} periods by

$$\bar{x} = (x_1^n, x_2^n, \ldots, x_m^n) .$$

We will introduce a decision vector $\bar{\theta} = (\theta_1^n, \theta_2^n, \ldots, \theta_m^n)$ where θ_i^n is the number of labourers assigned to machine center i in the n^{th} period. We shall say that decision vector $\bar{\theta}$ which satisfies the following constraints (a) through (e) belongs to the set U of admissible decision vectors.

a) If $\sum_{i=1}^{m} \theta_i^n < L$ then there cannot exist an i such that $\theta_i^n < f_i$ and $x_i^n > 0$ for $0 < n \leq N$ where N is total number of periods.

b) $\theta_i^n = 0$ whenever $x_i^n = 0$ for $i = 1, 2, \ldots, m$; $0 < n \leq N$.

c) θ_i^n = an integer for $i = 1, 2, \ldots, m$ and $0 < n \leq N$.

d) $0 \leq \theta_i^n \leq f_i$, $i = 1, 2, \ldots, m$ and $0 < n \leq N$.

e) $\sum_{i=1}^{i=m} \theta_i^n \leq L$ for $0 < n \leq N$.

The meaning of constraint (a) is that as many as possible of the labourers will be used in any given period. Constraint (b) states that labourers are to be assigned only to machine centers that have work to be performed in any given period. Constraint (c) is an indication of indivisibility of a single labourer. Constraint (d) signifies the limitations of the machine centers to absorb labour productivity. Constraint (e) assures that the total size of the labour force is not exceeded.

The main objective of the problem is to minimize

$$\sum_{n=1}^{n=N} \sum_{i=1}^{i=m} k_i x_i^n \; .$$

6.1 AN ALGORITHM BASED ON THE DISCRETE MAXIMUM PRINCIPLE

The performance equations are given by

$$x_i^n = x_i^{n-1} + p_{0i}^n \lambda + \sum_{\substack{j=1 \\ j\neq i}}^{j=m} p_{ji}^n \mu_j \theta_j^n - \mu_i \theta_i^n, \quad i = 1, 2, \ldots, m, \tag{77}$$

$$x_i^0 = \alpha$$

where

p_{ij}^{n} = represents the transition of work units from machine center i to j in the n^{th} period. This is equal to one if work is transferred and zero otherwise.

Similarly

p_{0i}^{n} = represents the transition of work units from outside to machine center i in the n^{th} period. This is equal to one if work is transferred and zero otherwise.

The second, third and the fourth terms on right hand side of equation (77) represent charges in queue length caused by work units arriving from outside the system, work units arriving from other machine centers and work units completed and departing for subsequent processing. Now let us define a new state variable x_{m+1}^{n} to represent cost.

$$x_{m+1}^{n} = \text{Total cost up to and including } n^{th} \text{ period}$$

$$= x_{m+1}^{n-1} + \sum_{i=1}^{m} k_i x_i^n \quad . \tag{78}$$

The objective function to be minimized is

$$S = \sum_{i=1}^{m} c_i x_i^N + c_{m+1} x_{m+1}^N = x_{m+1}^N \quad . \tag{79}$$

So that

$$c_i = 0, \quad i = 1, 2, \ldots, m,$$

$$c_{m+1} = 1 \ .$$

The Hamiltonian function and adjoint variables are written as follows.

$$H^n = \sum_{i=1}^{i=m} z_i^n x_i^n + z_{m+1}^n x_{m+1}^n$$

$$= \sum_{i=1}^{i=m} z_i^n \left(x_i^{n-1} + p_{0i}^n \lambda + \sum_{\substack{j=1 \\ j \neq i}}^{i=m} p_{ji}^{n-1} \mu_j \theta_j^{n-1} - \mu_i \theta_i^n\right)$$

$$+ z_{m+1}^n \left(x_{m+1}^{n-1} + \sum_{i=1}^{i=m} k_i x_i^n\right), \tag{80}$$

$$z_i^{n-1} = \frac{\partial H^n}{\partial x_i^{n-1}} = z_i^n + z_{m+1}^n k_i \ ,$$

$i = 1, 2, \ldots, m,$

$n = 1, 2, \ldots, N,$

$z_i^N = 0, \quad i = 1, 2, \ldots, m,$

$$z_{m+1}^{n-1} = \frac{\partial H^n}{\partial x_{m+1}^{n-1}} = z_{m+1}^n \ ,$$

$n = 1, 2, \ldots, N,$

$z_{m+1}^N = c_{m+1} = 1,$

so that

$$z_{m+1}^n = 1, \quad n = 1, 2, \ldots, N.$$

Substituting $z_{m+1}^n = 1$ in H^n and rearranging the terms, we obtain

$$H^n = \sum_{i=1}^{i=m} z_i^n x_i^{n-1} + \sum_{i=1}^{i=m} z_i^n p_{0i}^n \lambda + x_{m+1}^{n-1} +$$

$$\sum_{i=1}^{i=m} k_i x_i^{n-1} + \sum_{i=1}^{i=m} (z_i^n) \sum_{\substack{j=1 \\ j \neq i}}^{m} p_{ji}^{n-1} \mu_j \theta_j^{n-1} - \sum_{i=1}^{i=m} z_i^n \mu_i \theta_i^n +$$

$$\sum_{i=1}^{i=m} k_i p_{0i}^n \lambda + \sum_{i=1}^{i=m} k_i \sum_{\substack{j=1 \\ j \neq i}}^{j=m} p_{ji}^{n-1} \mu_j \theta_j^{n-1} - \sum_{i=1}^{i=m} k_i \mu_i \theta_i^n \ . \qquad (81)$$

S is a minimum when H^n is a minimum. From equation (81), we find that H^n is a linear with respect to θ_i^n . Hence $\bar{\theta}_i^n$ which makes H^n a minimum lies on the boundary of set U of decision vectors. In equation (81) z_i^n, x_i^{n-1} , x_{m+1}^{n-1}, p_{0i}^n, p_{ji}^{n-1} , k_i , λ , μ_i and μ_j are constants. Therefore the variable portion of H^n, H_v^n, can be written as

$$H_v^n = - \sum_{i=1}^{i=m} z_i^n \mu_i \theta_i^n - \sum_{i=1}^{i=m} k_i \mu_i \theta_i^n$$

$$= - \sum_{i=1}^{i=m} (z_i^n + k_i) \mu_i \theta_i^n$$

$$= - \sum_{i=1}^{i=m} z_i^{n-1} \mu_i \theta_i^n \ . \qquad (82)$$

Obviously S is a minimum when H_v^n is a minimum. From equation (82), it can be seen that H_v^n is a minimum when θ_i^n is a maximum. Physical interpretation of maximizing θ_i^n is to allot as much of the labor force as possible to fill machine center for which $x_i^n > 0$. In doing so, we should

evolve some criteria based on which the labour force is divided among the facilities for which $x_i^n > 0$. For this purpose, we will define two more terms. Let

μ_i^n = service rate in work units per period for the machine centre to which work being processed at machine centre i in the n^{th} period is flowing for its next process.

Similarly

k_i^n = Inventory changes per unit per period for the next machine centre line.

f_i^n = Number of machines in the next machine centre in line.

We know

$$z_i^{n-1} = z_i^n + k_i \ ,$$

so that

$$z_i^{N-1} = z_i^N + k_i$$

$$= 0 + k_i \ ,$$

$$z_i^{N-2} = k_i + k_i = 2k_i$$

and so on.

In general

$$z_i^{n-1} = \beta k_i \tag{83}$$

where

β = A non-negative constant quantity.

Substituting equation (83) in H_v^n, we have

$$H_v^n = - \sum_{i=1}^{i=m} \beta k_i \mu_i \theta_i^n .$$

In order to minimize H_v^n, allot maximum θ_i^n to the machine centre for which

a) $x_i^n > 0$,

b) $k_i \mu_i$ is a maximum .

But there is one danger, namely that if for the next machine in the line, the service rate is low (in other words μ_i^n is low) and k_i^n is high, this decision will not constitute an optimum policy. In order to avoid this undesirable situation, let us calculate a time dependent priority Π_i^n given by

$$\Pi_i^n = (f_i^n \mu_i^n k_i - f_i \mu_i k_i^n) \ , \quad i = 1, 2, \ldots, m.$$

Optimum policy is to allot in any period n, as many of L labourers as possible to fill machine centres for which $x_i^n > 0$ in order of decreasing values of Π_i^n .

6.2 EXAMPLE 7. A SIMPLE NUMERICAL EXAMPLE

Let us consider three machine centres, so that m = 3. Work pieces are processed first on machine centre 1, then on 2 and finally on 3. There are 7 identical machines in machine centre 1, 16 machines in machine centre 2 and 7 machines in machine centre 3.

Work pieces arrive at the rate of 60 units/hour. Service rate for each machine for machine centre 1 is 10 units per hour, for centre 2, 5 units

per hour and for centre 3, 12 units per hour. A tote box is used to carry the completed work pieces every hour and has a capacity of 90 units.

The inventory charges are \$1/unit/hour for machine centre 1, \$0.50/unit/hour for machine centre 2 and \$0.75/unit/hour for machine centre 3. A maximum of 20 labourers are available. Each labourer is assumed to be equally competent to work on any machine centre. Assuming that there is no initial in-process inventory, determine how many labourers should be assigned to each machine centre every hour, to minimize the in-process inventory cost for a time-span of 5 hours, assuming that the inventory cost for the finished products is \$0.60 per unit per hour.

6.3 SOLUTION BY THE DISCRETE MAXIMUM PRINCIPLE

The following values are given.

$$n = 3,\ N = 5,\ L = 20,$$

$$\lambda = 60 \text{ units/hour},\ \mu_1 = 10 \text{ units/hour},$$

$$\mu_2 = 5 \text{ units/hour},\ \mu_3 = 12 \text{ units/hour},$$

$$f_1 = 7, \qquad f_2 = 16, \qquad f_3 = 7,$$

$$k_1 = \$1, \quad k_2 = \$0.50, \quad k_3 = \$0.75, \quad k_4 = \$0.60 .$$

Let us denote the capacity of the tote box by $f_4\mu_4$, then $f_4\mu_4 = 90$.

Let us first calculate the time dependent priority for each machine centre. In this case it may be noted

$$k_i^n = k_{i+1} \text{ and } \mu_i^n = \mu_{i+1}, \quad n = 1, 2, 3, 4, 5,$$
$$i = 1, 2, 3,$$

$$\Pi_1^n = f_1^n\mu_1^n k_1 - f_1\mu_1 k_1^n$$

$$= (16 \times 5 \times 1 - 7 \times 10 \times 0.5) = 45,$$

$$\Pi_2^n = (7 \times 12 \times 0.5 - 16 \times 5 \times 0.75) = -18,$$

$$\Pi_3^n = (90 \times 0.75 - 7 \times 12 \times 0.6) = 17.1,$$

for $n = 1, 2, 3, 4, 5$.

Hence we note

$$\Pi_1^n > \Pi_3^n > \Pi_2^n \quad .$$

Therefore, if there is queue length of work pieces at all the three machine centres, the maximum number of labourers is assigned to machine centre 1, then to 3 and to 2.

The assignment is shown in the following table.

No. of hour	No. of Labourers assigned to m/c. centre			Queue length in units at m/c centre			Inventory Cost $	Cumulative Inventory Cost $
	1	2	3	1	2	3		
1	6	0	0	60	0	0	0	0
2	6	12	0	60	60	0	0	0
3	6	9	5	60	60	60	7.50	7.50
4	6	10	4	60	75	45	12.50	20.00
5	6	10	4	60	85	50	19.0	39.00

Hence total minimum in-process inventory

cost during the time span of 5-hours = \$39.00

REFERENCES

1. Bellman, R.
"Dynamic Programming," Princeton University Press, Princeton, New Jersey (1957).

2. Fan, L. T. and C. S. Wang.
"The Discrete Maximum Principle-A Study of Multistage Systems Optimization," Wiley, New York, 1964.

3. Hwang, C. L. and F. A. Tillman and L. T. Fan.
"Optimization of Production Scheduling." accepted for publication in International Journal of Production Research, February, 1967.

4. Hwang, C. L.
Unpublished Lecture Notes, Kansas State University, (1966).

5. Hwang, C. L. and L. T. Fan.
"The Application of the Maximum Principle to Industrial and Management Systems," Journal of Industrial Engineering, Vol. 17, pp 589-590 (Nov. 1966).

6. Hwang, C. L., S. K. Chen, G. F. Schrader, J. M. Panchal and L. T. Fan.
"Application of The Maximum Principle to Transportation Problems with Linear and Non-linear Cost Functions," Special Report No. 75, Engineering Experiment Station, Kansas State University, (1966).

7. Nelson, R. T.
"Labour Assignment as a Dynamic Control Problem," Operations Research, Vol. 14, pp. 369-375 (1966).

8. Zangwill, W. I.
"A Deterministic Multiproduct Multifacility Production and Inventory Model," Operations Research, Vol. 14, pp. 486-507 (1966).

9. Zangwill, W. I.

"A Deterministic Multiperiod Production Scheduling Model with Backlogging," Management Science, Vol. 13, pp. 105-120 (1966).

ACKNOWLEDGEMENTS

The author wishes to express his keen gratitude to the following persons: Dr. C. L. Hwang, major professor, for his sincere guidance, valuable suggestions and cooperation in preparing this thesis; Dr. F. A. Tillman and Dr. L. E. Grosh for their initial guidance and cooperation; and Dr. M. V. Yadin of Department of Statistics for the interest shown by him in and valuable comments on this thesis.

OPTIMIZATION OF MULTIFACILITY
PRODUCTION SCHEDULING

by

JAYAWANTH DATTATREYARAO BANTWAL

B.Sc Engineering Kerala University
Kerala, India, 1964

AN ABSTRACT OF A MASTER' THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Industrial Engineering

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1967

ABSTRACT

The objective of this report is to present the study made by Zangwill dealing with a deterministic multiproduct, multifacility production planning and inventory model and that by Nelson concerned with labour assignment as a dynamic control problem. The optimization technique employed by Zangwill in arriving at an optimal production schedule is the well known dynamic programming. In this report, the same solution has been obtained by a discrete version of the maximum principle. Nelson has optimized the labour assignment in a labour and machine limited production system by the continuous maximum principle. In this report, the same problem for discrete time intervals is studied by the discrete maximum principle.

The deterministic multiproduct, multifacility, multiperiod production planning and inventory model developed by Zangwill is essentially a linking together of several single facility models. The model considers concave production costs which can depend upon production in several different facilities and piecewise concave inventory costs. The optimization problem consists in determining how much each facility in the network should produce in each period for the multiperiods so that the total production and inventory cost is minimized.

The production scheduling discussed in this report essentially refers to the parallel facility case, series facility case and multiperiod production planning model with backlog of demand. In the parallel facility situation, there are more than one facility in parallel. Each facility produces one product for which the demand for the next n periods is known. In the series facility case there are more than one facility connected in series. Each facility supplies input to the next facility in the line. The last facility in the line supplies the market. Next the multiperiod

production planning model with backlog of demand is presented. For all these three cases, dynamic programming algorithms are presented. Examples 1, 3, 4, 5 and 6 demonstrate the usefulness of these algorithms. These examples are also solved by the discrete maximum principle. Example 2 involves the parallel facility case with non-linear cost function and is solved by the discrete maximum principle.

The last section of this report is devoted to the discussion of a labour assignment as a dynamic control problem. The criterion employed here is to minimize the in-process inventory cost of the work pieces which are processed on different machines in a definite order. The original model developed by Nelson assumes continuous arrival of work pieces at the machine centre and hence the continuous maximum principle is employed in optimizing the labour assignment. The model considered in this report assumes the arrival of work pieces at discrete time intervals. Hence the discrete maximum principle is employed in optimizing the labour assignment. A simple numerical example is developed to demonstrate the applicability of the algorithm.